

**Representation of the Resonance
of a
Relativistic Quantum Field Theoretical Lee-Friedrichs Model
in
Lax-Phillips Scattering Theory**

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Abstract: The quantum mechanical description of the evolution of an unstable system defined initially as a state in a Hilbert space at a given time does not provide a semigroup (exponential) decay law. The Wigner-Weisskopf survival amplitude, describing reversible quantum transitions, may be dominated by exponential type decay in pole approximation at times not too short or too long, but, in the two channel case, for example, the pole residues are not orthogonal, and the evolution does not correspond to a semigroup (experiments on the decay of the neutral K -meson system strongly support the semigroup evolution postulated by Lee, Oehme and Yang, and Yang and Wu). The scattering theory of Lax and Phillips, originally developed for classical wave equations, has been recently extended to the description of the evolution of resonant states in the framework of quantum theory. The resulting evolution law of the unstable system is that of a semigroup, and the resonant state is a well-defined function in the Lax-Phillips Hilbert space. In this paper we apply this theory to a relativistically covariant quantum field theoretical form of the (soluble) Lee model. We construct the translation representations with the help of the wave operators, and show that the resulting Lax-Phillips S -matrix is an inner function (the Lax-Phillips theory is essentially a theory of translation invariant subspaces). In the special case that the S -matrix is a rational inner function, we obtain the resonant state explicitly and analyze its particle (V, N, θ) content. If there is an exponential bound, the general case differs only by a so-called trivial inner factor, which does not change the complex spectrum, but may affect the wave function of the resonant state.

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1. Introduction.

The theory of Lax and Phillips¹ (1967), originally developed for the description of resonances in electromagnetic or acoustic scattering phenomena, has been used as a framework for the construction of a description of irreversible resonant phenomena in the quantum theory²⁻⁵ (which we will refer to as the quantum Lax-Phillips theory). This leads to a time evolution of resonant states which is of semigroup type, i.e., essentially exponential decay. Semigroup evolution is necessarily a property of irreversible processes⁶. It appears experimentally that elementary particle decay, to a high degree of accuracy, follows a semigroup law, and hence such processes seem to be irreversible.

The theory of Weisskopf and Wigner⁷, which is based on the definition of the survival amplitude of the initial state ϕ (associated with the unstable system) as the scalar product of that state with the unitarily evolved state,

$$(\phi, e^{-iHt}\phi) \quad (1.1)$$

cannot have exact exponential behavior⁸. One can easily generalize this construction to the problem of more than one resonance^{9,10}. If P is the projection operator into the subspace of initial states (N -dimensional for N resonances), the reduced evolution operator is given by

$$Pe^{-iHt}P. \quad (1.1')$$

This operator cannot be an element of a semigroup.⁸

Experiments on the decay of the neutral K -meson system¹¹ show clearly that the phenomenological description of Lee, Oehme and Yang¹², and Wu and Yang¹³, by means of a 2×2 effective Hamiltonian which corresponds to an exact semigroup evolution of the unstable system, provides a very accurate description of the data. It can be proved that the Wigner-Weisskopf theory cannot provide a semigroup evolution law⁸ and, thus, an effective 2×2 Hamiltonian cannot emerge in the framework of this theory. Furthermore, it has been shown, using estimates based on the quantum mechanical Lee-Friedrichs model¹⁴, that the experimental results appear to rule out the application of the Wigner-Weisskopf theory to the decay of the neutral K -meson system. While the exponential decay law can be exhibited explicitly in terms of a Gel'fand triple¹⁵, the representation of the resonant state in this framework is in a Banach space which does not coincide with the quantum mechanical Hilbert space, and does not have the properties of a Hilbert space, such as scalar products and the possibility of calculating expectation values. One cannot compute physical properties other than the lifetimes in this way.

The quantum Lax-Phillips theory provides the possibility of constructing a fundamental theoretical description of the resonant system which has exact semigroup evolution, and represents the resonance as a *state in the Hilbert space*. In the following, we describe briefly the structure of this theory, and give some physical interpretation for the states of the Lax-Phillips Hilbert space.

The Lax-Phillips theory is defined in a Hilbert space $\overline{\mathcal{H}}$ of states which contains two distinguished subspaces, \mathcal{D}_{\pm} , called “outgoing” and “incoming”. There is a unitary evolution law which we denote by $U(\tau)$, for which these subspaces are invariant in the

following sense:

$$\begin{aligned} U(\tau)\mathcal{D}_+ &\subset \mathcal{D}_+ & \tau \geq 0 \\ U(\tau)\mathcal{D}_- &\subset \mathcal{D}_- & \tau \leq 0 \end{aligned} \quad (1.2)$$

The translates of \mathcal{D}_\pm under $U(\tau)$ are dense, i.e.,

$$\overline{\bigcup_{\tau} U(\tau)\mathcal{D}_\pm} = \overline{\mathcal{H}} \quad (1.3)$$

and the asymptotic property

$$\bigcap_{\tau} U(\tau)\mathcal{D}_\pm = \emptyset \quad (1.4)$$

is assumed. It follows from these properties that

$$Z(\tau) = P_+ U(\tau) P_-, \quad (1.5)$$

where P_\pm are projections into the subspaces orthogonal to \mathcal{D}_\pm , is a strongly contractive semigroup¹, i.e.,

$$Z(\tau_1)Z(\tau_2) = Z(\tau_1 + \tau_2) \quad (1.6)$$

for τ_1, τ_2 positive, and $\|Z(\tau)\| \rightarrow 0$ for $\tau \rightarrow 0$. It follows from (1.2) that $Z(\tau)$ takes the subspace \mathcal{K} , the orthogonal complement of \mathcal{D}_\pm in $\overline{\mathcal{H}}$ (associated with the resonances in the Lax-Phillips theory), into itself¹, i.e.,

$$Z(\tau) = P_{\mathcal{K}} U(\tau) P_{\mathcal{K}}. \quad (1.7)$$

The relation (1.7) is of the same structure as (1.1'); there is, as we shall see in the following, an essential difference in the way that the subspaces associated with resonances are defined. The argument that (1.1') cannot form a semigroup is not valid³ for (1.7).

There is a theorem of Sinai¹⁶ which affirms that a Hilbert space with the properties that there are distinguished subspaces satisfying, with a given law of evolution $U(\tau)$, the properties (1.2), (1.3), (1.4) has a foliation into a one-parameter (which we shall denote as s) family of isomorphic Hilbert spaces, which are called auxiliary Hilbert spaces, \mathcal{H}_s for which

$$\overline{\mathcal{H}} = \int_{\oplus} \mathcal{H}_s. \quad (1.8)$$

Representing these spaces in terms of square-integrable functions, we define the norm in the direct integral space (we use Lebesgue measure) as

$$\|f\|^2 = \int_{-\infty}^{\infty} ds \|f_s\|_H^2, \quad (1.9)$$

where $f \in \overline{\mathcal{H}}$ represents a vector in $\overline{\mathcal{H}}$ in terms of a function in the L^2 function space $\overline{\mathcal{H}} = L^2(-\infty, \infty, H)$; f_s is an element of H , the L^2 function space (which we shall call the *auxiliary space*) representing \mathcal{H}_s for any s [we shall not add in what follows a subscript to

the norm or scalar product symbols for scalar products of elements of the auxiliary Hilbert space associated to a point s on the foliation axis].

The Sinai theorem furthermore asserts that there are representations for which the action of the full evolution group $U(\tau)$ on $L^2(-\infty, \infty; H)$ is translation by τ units. Given D_{\pm} (the subspaces of L^2 functions representing \mathcal{D}_{\pm}), there is such a representation, called the *incoming representation*¹, for which the set of all functions in D_- have support in $(-\infty, 0)$ and constitute the subspace $L^2(-\infty, 0; H)$ of $L^2(-\infty, \infty; H)$; there is another representation, called the *outgoing representation*, for which functions in D_+ have support in $(0, \infty)$ and constitute the subspace $L^2(0, \infty; H)$ of $L^2(-\infty, \infty; H)$. The fact that $Z(\tau)$ in Eq. (1.7) is a semigroup is a consequence of the definition of the subspaces D_{\pm} in terms of support properties on intervals along the foliation axis in the *outgoing* and *incoming* translation representations respectively. The non self-adjoint character of the generator of the semigroup $Z(\tau)$ is a consequence of this structure.

Lax and Phillips¹ show that there are unitary operators W_{\pm} , called wave operators, which map elements in $\overline{\mathcal{H}}$, respectively, to these representations. They define an S -matrix,

$$S = W_+ W_-^{-1} \quad (1.10)$$

which connects the incoming to the outgoing representations; it is unitary, commutes with translations, and maps $L^2(-\infty, 0; H)$ into itself. Since S commutes with translations, it is diagonal in Fourier (spectral) representation. As pointed out by Lax and Phillips¹, according to a special case of a theorem of Fourès and Segal¹⁷, an operator with these properties can be represented as a multiplicative operator-valued function $\mathcal{S}(\sigma)$ which maps H into H , and satisfies the following conditions:

- (a) $\mathcal{S}(\sigma)$ is the boundary value of an operator-valued function $\mathcal{S}(z)$ analytic for $\text{Im} z > 0$.
- (b) $\|\mathcal{S}(z)\| \leq 1$ for all z with $\text{Im} z > 0$.
- (c) $\mathcal{S}(\sigma)$ is unitary for almost all real σ .

An operator with these properties is known as an inner function¹⁸; such operators arise in the study of shift invariant subspaces, the essential mathematical content of the Lax-Phillips theory. The singularities of this S -matrix, in what we shall define as the *spectral representation* (defined in terms of the Fourier transform on the foliation variable s), correspond to the spectrum of the generator of the semigroup characterizing the evolution of the unstable system.

In the framework of quantum theory, one may identify the Hilbert space \mathcal{H} with a space of physical states, and the variable τ with the laboratory time (the semigroup evolution is observed in the laboratory according to this time). The representation of this space in terms of the foliated L^2 space $\overline{\mathcal{H}}$ provides a natural probabilistic interpretation for the auxiliary spaces associated with each value of the foliation variable s , i.e., the quantity $\|f_s\|^2$ corresponds to the probability density for the system to be found in the neighborhood of s . For example, consider an operator A defined on $\overline{\mathcal{H}}$ which acts pointwise, i.e., contains no shift along the foliation. Such an operator can be represented as a direct integral

$$A = \int_{\oplus} A_s. \quad (1.11)$$

It produces a map of the auxiliary space H into H for each value of s , and thus, if it is self-adjoint, A_s may act as an observable in a quantum theory associated to the point s .⁴ The expectation value of A_s in a state in this Hilbert space defined by the vector ψ_s , the component of $\psi \in \overline{H}$ in the auxiliary space at s , is

$$\langle A_s \rangle_s = \frac{(\psi_s, A_s \psi_s)}{\|\psi_s\|^2}. \quad (1.12)$$

Taking into account the *a priori* probability density $\|\psi_s\|^2$ that the system is found at this point on the foliation axis, we see that the expectation value of A in \overline{H} is

$$\langle A \rangle = \int ds \langle A_s \rangle_s \|\psi_s\|^2 = \int ds (\psi_s, A_s \psi_s), \quad (1.13)$$

the direct integral representation of $(\psi, A\psi)$.

As we have remarked above, in the translation representations for $U(\tau)$ the foliation variable s is shifted (this shift, for sufficiently large $|\tau|$, induces the transition of the state into the subspaces \mathcal{D}_\pm). It follows that s may be identified as an intrinsic time associated with the evolution of the state; since it is a variable of the measure space of the Hilbert space $\overline{\mathcal{H}}$, this quantity itself has the meaning of a quantum variable.

We are presented here with the notion of a virtual history. To understand this idea, suppose that at a given time τ_0 , the function which represents the state has some distribution $\|\psi_{s\tau_0}\|^2$. This distribution provides an *a priori* probability that the system would be found at time s (greater or less than τ_0), if the experiment were performed at time s corresponding to $\tau = s$ on the laboratory clock. The state of the system therefore contains information on the structure of the *history* of the system as it is inferred at τ_0 .

We shall assume the existence of a unitary evolution on the Hilbert space $\overline{\mathcal{H}}$, and that for

$$U(\tau) = e^{-iK\tau}, \quad (1.14)$$

the generator K can be decomposed as

$$K = K_0 + V \quad (1.15)$$

in terms of an unperturbed operator K_0 with spectrum $(-\infty, \infty)$ and a perturbation V , under which this spectrum is stable. We shall, furthermore, assume that wave operators exist, defined on some dense set, as

$$\Omega_\pm = \lim_{\tau \rightarrow \pm\infty} e^{iK\tau} e^{-iK_0\tau}. \quad (1.16)$$

In the soluble model that we shall treat as an example in this paper, the existence of the wave operators is assured.

With the help of the wave operators, we can define translation representations for $U(\tau)$. The translation representation for K_0 is defined by the property

$${}_0\langle s, \alpha | e^{-iK_0\tau} f \rangle = {}_0\langle s - \tau, \alpha | f \rangle, \quad (1.17)$$

where α corresponds to a label for the basis of the auxiliary space. Noting that

$$K\Omega_{\pm} = \Omega_{\pm}K_0 \quad (1.18)$$

we see that

$${}_{in}^{out}\langle s, \alpha | e^{-iK\tau} f \rangle = {}_{in}^{out}\langle s - \tau, \alpha | f \rangle, \quad (1.19)$$

where

$${}_{in}^{out}\langle s, \alpha | f \rangle = {}_0\langle s, \alpha | \Omega_{\pm}^{\dagger} f \rangle \quad (1.20)$$

It will be convenient to work in terms of the Fourier transform of the *in* and *out* translation representations; we shall call these the *in* and *out spectral* representations, *i.e.*,

$${}_{in}^{out}\langle \sigma, \alpha | f \rangle = \int_{-\infty}^{\infty} e^{-\sigma s} {}_{in}^{out}\langle s, \alpha | f \rangle. \quad (1.21)$$

In these representations, (1.20) is

$${}_{in}^{out}\langle \sigma, \alpha | f \rangle = {}_0\langle \sigma, \alpha | \Omega_{\pm}^{\dagger} f \rangle \quad (1.22)$$

and (1.19) becomes

$${}_{in}^{out}\langle \sigma, \alpha | e^{-iK\tau} f \rangle = e^{-i\sigma\tau} {}_{in}^{out}\langle \sigma, \alpha | f \rangle. \quad (1.23)$$

Eq. (1.17) becomes, under Fourier transform

$${}_0\langle \sigma, \alpha | e^{-iK_0\tau} f \rangle = e^{-i\sigma\tau} {}_0\langle \sigma, \alpha | f \rangle. \quad (1.24)$$

For f in the domain of K_0 , (1.23) implies that

$${}_0\langle \sigma, \alpha | K_0 f \rangle = \sigma {}_0\langle \sigma, \alpha | f \rangle. \quad (1.25)$$

With the solution of (1.25), and the wave operators, the *in* and *out* spectral representations of a vector f can be constructed from (1.24).

We are now in a position to construct the subspaces \mathcal{D}_{\pm} , which are not given, *a priori*, in the Lax-Phillips quantum theory. Identifying ${}_{out}\langle s, \alpha | f \rangle$ with the *outgoing* translation representation, we shall define D_+ as the set of functions with support in $(0, \infty)$ in this representation. Similarly, identifying ${}_{in}\langle s, \alpha | f \rangle$ with the *incoming* translation representation, we shall define D_- as the set of functions with support in $(-\infty, 0)$ in this representation. The corresponding elements of \mathcal{H} constitute the subspaces \mathcal{D}_{\pm} . By construction, \mathcal{D}_{\pm} have the required invariance properties under the action of $U(\tau)$.

The *outgoing spectral representation* of a vector $g \in \mathcal{H}$ is

$$\begin{aligned} {}_{out}\langle \sigma \alpha | g \rangle &= {}_0\langle \sigma \alpha | \Omega_+^{-1} g \rangle = \int d\sigma' \sum_{\alpha'} {}_0\langle \sigma \alpha | \mathbf{S} | \sigma' \alpha' \rangle {}_0\langle \sigma' \alpha' | \Omega_-^{-1} g \rangle \\ &= \int d\sigma' \sum_{\alpha'} {}_0\langle \sigma \alpha | \mathbf{S} | \sigma' \alpha' \rangle {}_{in}\langle \sigma' \alpha' | g \rangle, \end{aligned} \quad (1.26)$$

where we call

$$\mathbf{S} = \Omega_+^{-1} \Omega_- . \quad (1.27)$$

the quantum Lax-Phillips S -operator. We see that the kernel ${}_0\langle\sigma\alpha|\mathbf{S}|\sigma'\alpha'\rangle_0$ maps the incoming to outgoing spectral representations. Since \mathbf{S} commutes with K_0 , it follows that

$${}_0\langle\sigma\alpha|\mathbf{S}|\sigma'\alpha'\rangle_0 = \delta(\sigma - \sigma') S^{\alpha\alpha'}(\sigma) \quad (1.28)$$

It follows from (1.16) and (1.22), in the standard way¹⁹, that

$${}_0\langle\sigma\alpha|\mathbf{S}|\sigma'\alpha'\rangle_0 = \lim_{\epsilon \rightarrow 0} \delta(\sigma - \sigma') \{ \delta^{\alpha\alpha'} - 2\pi i {}_0\langle\sigma\alpha|\mathbf{T}(\sigma + i\epsilon)|\sigma'\alpha'\rangle_0 \}, \quad (1.29)$$

where

$$\mathbf{T}(z) = V + VG(z)V = V + VG_0(z)\mathbf{T}(z). \quad (1.30)$$

We remark that, by this construction, $S^{\alpha\alpha'}(\sigma)$ is *analytic in the upper half plane* in σ . The Lax-Phillips S -matrix¹ is given by the inverse Fourier transform,

$$S = \{ {}_0\langle s\alpha|\mathbf{S}|s'\alpha'\rangle_0 \}; \quad (1.31)$$

this operator clearly commutes with translations.

From (1.29) it follows that the property (a) above is true. Property (c), unitarity for real σ , is equivalent to asymptotic completeness, a property which is stronger than the existence of wave operators. For the relativistic Lee model, which we shall treat in this paper, this condition is satisfied. In the model that we shall study here, we shall see that there is a wide class of potentials V for which the operator $S(\sigma)$ satisfies the property (b) specified above.

In the next section, we review briefly the structure of the relativistic Lee model¹⁹, and construct explicitly the Lax-Phillips spectral representations and S -matrix. Introducing auxiliary space variables, we then characterize the properties of the finite rank Lee model potential which assure that the S -matrix is an inner function, *i.e.*, that property (b) listed above is satisfied.

2. Relativistic Lee-Friedrichs Model

In this section, we define the relativistic Lee model¹⁹ in terms of bosonic quantum fields on spacetime ($x \equiv x^\mu$). These fields evolve with an invariant evolution parameter²⁰ τ (which we identify here with the evolution parameter of the Lax-Phillips theory discussed above); at equal τ , they satisfy the commutation relations (with ψ^\dagger as the canonical conjugate field to ψ ; the fields ψ , which satisfy first order evolution equations as for nonrelativistic Schrödinger fields, are just annihilation operators)

$$[\psi_\tau(x), \psi_\tau^\dagger(x')] = \delta^4(x - x'). \quad (2.1)$$

We remark that Antoniou, *et al*²¹, have constructed a relativistic Lee model of a somewhat different type; their field equation is second order in the derivative with respect to the variable conjugate to the mass.

In momentum space, for which

$$\psi_\tau(p) = \frac{1}{(2\pi)^2} \int d^4x e^{-ip_\mu x^\mu} \psi_\tau(x), \quad (2.2)$$

this relation becomes

$$[\psi_\tau(p), \psi_\tau^\dagger(p')] = \delta^4(p - p'). \quad (2.3)$$

The manifestly covariant spacetime structure of these fields is admissible when E, \mathbf{p} are not *a priori* constrained by a sharp mass-shell relation. In the mass-shell limit (for which the variations in m^2 defined by $E^2 - \mathbf{p}^2$ are small), multiplying both sides of (2.3) by $\Delta E = \Delta m^2/2E$, one obtains the usual commutation relations for on-shell fields,

$$[\tilde{\psi}_\tau(\mathbf{p}), \tilde{\psi}^\dagger(\mathbf{p}')] = 2E\delta^3(\mathbf{p} - \mathbf{p}'), \quad (2.4)$$

where $\tilde{\psi}(\mathbf{p}) = \sqrt{\Delta m^2} \psi(p)$. The generator of evolution

$$K = K_0 + V \quad (2.5)$$

for which the Heisenberg picture fields are

$$\psi_\tau(p) = e^{iK\tau} \psi_0(p) e^{-iK\tau} \quad (2.6)$$

is given, in this model, as (we write $p^2 = p_\mu p^\mu$, $k^2 = k_\mu k^\mu$ in the following)

$$\begin{aligned} K_0 = \int d^4p \{ & \frac{p^2}{2M_V} b^\dagger(p) b(p) + \frac{p^2}{2M_N} a_N^\dagger(p) a_N(p) \} \\ & + \int d^4k \frac{k^2}{2M_\theta} a_\theta^\dagger(k) a_\theta(k) \end{aligned} \quad (2.7)$$

and

$$V = \int d^4p \int d^4k (f(k) b^\dagger(p) a_N(p - k) a_\theta(k) + f^*(k) b(p) a_N^\dagger(p - k) a_\theta^\dagger(k)), \quad (2.8)$$

describing the process $V \leftrightarrow N + \theta$. The fields $b(p)$, $a_N(p)$ and a_θ are annihilation operators for the V , N , and θ particles, respectively and M_v , M_N and M_θ are the mass parameters for the fields²².

The operators

$$\begin{aligned} Q_1 &= \int d^4p [b^\dagger(p) b(p) + a_N^\dagger(p) a_N(p)] \\ Q_2 &= \int d^4p [a_N^\dagger(p) a_N(p) - a_\theta^\dagger(p) a_\theta(p)] \end{aligned} \quad (2.9)$$

are conserved, enabling us to decompose the Fock space to sectors. We shall study the problem in the lowest sector $Q_1 = 1, Q_2 = 0$, for which there is just one V or one N and one θ . In this sector the generator of evolution K can be rewritten in the form

$$K = \int d^4p K^p = \int d^4p (K_0^p + V^p)$$

where

$$K_0^p = \frac{p^2}{2M_V} b^\dagger(p) b(p) + \int d^4k \left(\frac{(p-k)^2}{2M_N} + \frac{k^2}{2M_\theta} \right) a_N^\dagger(p-k) a_\theta^\dagger(k) a_\theta(k) a_N(p-k)$$

and

$$V^p = \int d^4k \left(f(k) b^\dagger(p) a_N(p-k) a_\theta(k) + f^*(k) b(p) a_N^\dagger(p-k) a_\theta^\dagger(k) \right)$$

In this form it is clear that both K and K_0 have a direct integral structure. This implies a similar structure for the wave operators Ω_\pm and the possibility of defining restricted wave operators Ω_\pm^p for each value of p . From the expression for K_0^p we see that $|V(p)\rangle = b^\dagger(p)|0\rangle$ can be regarded as a discrete eigenstate of K_0^p and, therefore, is annihilated by Ω_\pm^p . This, in turn, implies that $\Omega_\pm|V(p)\rangle = 0$ for every p (an explicit demonstration of this fact is given in appendix A).

In order to construct the Lax-Phillips incoming and outgoing spectral representations for the model presented here it is necessary, according to the discussion following equation (1.25), to obtain appropriate expressions for the wave operators Ω_\pm^\dagger and the solutions of equation (1.25).

We will consider first the problem of finding the generalized eigenvectors with spectrum $\{\sigma\}$ on $(-\infty, \infty)$, $|\sigma, \gamma\rangle_0$ of K_0 . The complete set of these states is decomposed into two subsets corresponding to the quantum numbers for states containing N and θ particles and those containing a V particle. These quantum numbers are denoted σ, α ($N + \theta$ states) and σ, β (V states) respectively. We have for the projections into these two subspaces,

$$\begin{aligned} |\sigma, \alpha\rangle_0 &= \int d^4p \int d^4k |N(p), \theta(k)\rangle \langle N(p), \theta(k)| \sigma, \alpha\rangle_0 \\ |\sigma, \beta\rangle_0 &= \int d^4p |V(p)\rangle \langle V(p)| \sigma, \beta\rangle_0 \end{aligned} \tag{2.10}$$

where we have denoted $|N(p), \theta(k)\rangle \equiv a_N^\dagger(p) a_\theta^\dagger(k) |0\rangle$ and $|V(p)\rangle \equiv b^\dagger(p) |0\rangle$. Defining

$$\begin{aligned} O_{p,k}^{\sigma,\alpha} &\equiv \langle N(p), \theta(k) | \sigma, \alpha\rangle_0 \\ O_p^{\sigma,\beta} &\equiv \langle V(p) | \sigma, \beta\rangle_0 \end{aligned} \tag{2.11}$$

we can write (2.10) as

$$\begin{aligned} |\sigma, \alpha\rangle_0 &= \int d^4p d^4k O_{p,k}^{\sigma,\alpha} |N(p), \theta(k)\rangle \\ |\sigma, \beta\rangle_0 &= \int d^4p O_p^{\sigma,\beta} |V(p)\rangle \end{aligned} \tag{2.12}$$

It follows from equations (1.25) and (2.12) that we must have

$$\begin{aligned} K_0 |\sigma, \alpha\rangle_0 &= (\omega_{N(p)} + \omega_{\theta(k)}) |\sigma, \alpha\rangle_0 = \sigma |\sigma, \alpha\rangle_0 \\ K_0 |\sigma, \beta\rangle_0 &= \omega_{V(p)} |\sigma, \beta\rangle_0 = \sigma |\sigma, \beta\rangle_0 \end{aligned}$$

where $\omega_{N(p)} = p^2/2M_N$, $\omega_{\theta(k)} = k^2/2M_\theta$ and $\omega_{N(p)} = p^2/2M_N$. The kinematic conditions imposed by these equations are satisfied if we set

$$\begin{aligned} O_{p,k}^{\sigma,\alpha} &= \delta(\sigma - \omega_{N(p)} - \omega_{\theta(k)}) \tilde{O}_{p,k}^{\sigma,\alpha} \\ O_p^{\sigma,\beta} &= \delta(\sigma - \omega_{V(p)}) \tilde{O}_p^{\sigma,\beta} \end{aligned} \quad (2.13)$$

A more detailed analysis of the structure of the matrix elements (2.11) can only be achieved with further knowledge of the nature of the variables α (defining the auxiliary Hilbert space). We will postpone the discussion of this point to later and remark here only that orthogonality and completeness requires that

$$\begin{aligned} \sum_{\alpha} \int d\sigma \left(O_{p,k}^{\sigma,\alpha} \right)^* O_{p',k'}^{\sigma,\alpha} &= \delta^4(p - p') \delta^4(k - k') \\ \int d^4p d^4k \left(O_{p,k}^{\sigma,\alpha} \right)^* O_{p,k}^{\sigma',\alpha'} &= \delta(\sigma - \sigma') \delta_{\alpha,\alpha'} \end{aligned} \quad (2.14)$$

$$\begin{aligned} \sum_{\beta} \int d\sigma \left(O_p^{\sigma,\beta} \right)^* O_{p'}^{\sigma,\beta} &= \delta^4(p - p') \\ \int d^4p \left(O_p^{\sigma,\beta} \right)^* O_p^{\sigma',\beta'} &= \delta(\sigma - \sigma') \delta_{\beta,\beta'} \end{aligned} \quad (2.14')$$

We turn now to the calculation of matrix elements of the wave operators and start by obtaining appropriate expressions for the following matrix elements of Ω_+

$$\langle V(p+k) | \Omega_+ | N(p), \theta(k) \rangle \quad \langle N(p'), \Theta(k') | \Omega_+ | N(p), \theta(k) \rangle$$

Equation (1.16) can be rewritten, following the standard manipulation²³, in the integral form

$$\Omega_+ = 1 + i \lim_{\epsilon \rightarrow 0} \int_0^{+\infty} U^\dagger(\tau) V U_0(\tau) e^{-\epsilon\tau} d\tau \quad (2.15)$$

where $U(\tau) = e^{-iK\tau}$, $U_0(\tau) = e^{-iK_0\tau}$. Using (2.7), we have

$$\begin{aligned} \Omega_+ | N(p), \theta(k) \rangle &= | N(p), \theta(k) \rangle + i \lim_{\epsilon \rightarrow 0} \int_0^{+\infty} d\tau U^\dagger(\tau) V U_0(\tau) e^{-\epsilon\tau} a_N^\dagger(p) a_\theta^\dagger(k) | 0 \rangle \\ &= | N(p), \theta(k) \rangle + i \lim_{\epsilon \rightarrow 0} \int_0^{+\infty} d\tau U^\dagger(\tau) V e^{-i(\omega_N(p) + \omega_\theta(k) - i\epsilon)\tau} a_N^\dagger(p) a_\theta^\dagger(k) | 0 \rangle \end{aligned} \quad (2.16)$$

Using (2.8) we find

$$V a_N^\dagger(p) a_\theta^\dagger(k) | 0 \rangle = f(k) b^\dagger(p+k) | 0 \rangle \quad (2.17)$$

Inserting (2.17) into (2.16) and changing the integration variable from τ to $-\tau$ it follows that

$$\Omega_+ | N(p), \theta(k) \rangle =$$

$$= |N(p), \theta(k)\rangle - i \lim_{\epsilon \rightarrow 0} \int_0^{-\infty} d\tau e^{i(\omega_N(p) + \omega_\theta(k) - i\epsilon)\tau} U(\tau) f(k) b^\dagger(p+k) |0\rangle. \quad (2.18)$$

In order to complete the evaluation of the integral in (2.18) we need to find the time evolution of a state ψ_0 under the action of $U(\tau)$

$$\psi(\tau) = e^{-iK\tau} \psi_0 \quad (2.19)$$

In the sector of the Fock space that we are considering, the state $\psi(\tau)$ at any time τ can be expanded as

$$\psi(\tau) = \int d^4q A(q, \tau) b^\dagger(q) |0\rangle + \int d^4p' \int d^4k' B(p', k', \tau) a_N^\dagger(p') a_\theta^\dagger(k') |0\rangle \quad (2.20).$$

In particular, we see that the initial conditions for the evolution in (2.18) are

$$B(p', k', 0) = 0 \quad A_{pk}(q, 0) = f(k) \delta^4(q - p - k) \quad (2.21)$$

The equations of evolution for the coefficients $A_{pk}(q, \tau)$ and $B(p', k', \tau)$ are then obtained from (2.19) and (2.20), *i.e.*,

$$\begin{aligned} i \frac{\partial A_{pk}(q, \tau)}{\partial \tau} &= \frac{q^2}{2M_V} A_{pk}(q, \tau) + \int d^4k' f(k') B(q - k', k', \tau) \\ i \frac{\partial B(p', k', \tau)}{\partial \tau} &= \left(\frac{p'^2}{2M_V} + \frac{k'^2}{2M_\theta} \right) B(p', k', \tau) + f^*(k') A_{pk}(p' + k', \tau) \end{aligned} \quad (2.22)$$

These equations can be solved algebraically¹⁹ by performing Laplace transforms and defining

$$\begin{aligned} \tilde{A}_{pk}(q, z) &= \int_0^{-\infty} e^{iz\tau} A_{pk}(q, \tau) d\tau \quad \text{Im} z < 0 \\ \tilde{B}(p', k', z) &= \int_0^{-\infty} e^{iz\tau} B(p', k', \tau) d\tau \quad \text{Im} z < 0 \end{aligned} \quad (2.23)$$

equations (2.22) are then transformed into

$$\begin{aligned} \left(z - \frac{q^2}{2M_V} \right) \tilde{A}_{pk}(q, z) &= i A_{pk}(q, 0) + \int d^4k' f(k') \tilde{B}(q - k', k', z) \\ \left(z - \frac{p'^2}{2M_N} - \frac{k'^2}{2M_\theta} \right) \tilde{B}(p', k', z) &= i B(p', k', 0) + f^*(k') \tilde{A}_{pk}(p' + k', z) \end{aligned} \quad (2.24)$$

Using the initial conditions (2.21) one obtains the following expressions for the Laplace transformed coefficients

$$\begin{aligned} \tilde{A}_{pk}(q, z) &= \frac{i A_{pk}(q, 0)}{h(q, z)} \\ \tilde{B}(p', k', z) &= \left(z - \frac{p'^2}{2M_N} - \frac{k'^2}{2M_\theta} \right)^{-1} f^*(k') \frac{i A_{pk}(p' + k', 0)}{h(p' + k', z)} \end{aligned} \quad (2.25)$$

where

$$h(q, z) \equiv z - \frac{q^2}{2M_V} - \int d^4k \frac{|f(k)|^2}{z - \frac{(q-k)^2}{2M_N} - \frac{k^2}{2M_\theta}} \quad (2.26)$$

The Laplace transform of $\psi(\tau)$ is then

$$\begin{aligned} \tilde{\psi}(z) &= i \int d^4q \frac{A_{pk}(q, 0)}{h(q, z)} b^\dagger(q) |0\rangle \\ &+ i \int d^4p' \int d^4k' \left(z - \frac{p'^2}{2M_N} - \frac{k'^2}{2M_\theta} \right)^{-1} \frac{f^*(k') A_{pk}(p' + k', 0)}{h(p' + k', z)} a_N^\dagger(p') a_\theta^\dagger(k') |0\rangle \end{aligned} \quad (2.27)$$

From (2.18), (2.21) and (2.27) we get

$$\begin{aligned} \Omega_+ |N(p), \theta(k)\rangle &= |N(p), \theta(k)\rangle + i \lim_{\epsilon \rightarrow 0} \left[-i \frac{f(k)}{h(p+k, \omega - i\epsilon)} b^\dagger(p+k) |0\rangle \right. \\ &\left. - i \int d^4p' \left(\omega - i\epsilon - \frac{p'^2}{2M_N} - \frac{p+k-p'^2}{2M_\theta} \right)^{-1} \frac{f^*(p+k-p') f(k)}{h(p+k, \omega - i\epsilon)} a_N^\dagger(p') a_\theta^\dagger(p+k-p') |0\rangle \right] \end{aligned} \quad (2.28)$$

where $\omega = \omega_N(p) + \omega_\theta(k)$. We can now evaluate the desired matrix elements of the wave operator

$$\langle V(\tilde{p}) | \Omega_+ | N(p), \theta(k) \rangle = \lim_{\epsilon \rightarrow 0} \delta^4(\tilde{p} - p - k) f(k) h^{-1}(\tilde{p}, \omega - i\epsilon) \quad (2.29)$$

and

$$\begin{aligned} \langle N(\tilde{p}), \theta(\tilde{k}) | \Omega_+ | N(p), \theta(k) \rangle &= \delta^4(\tilde{p} - p) \delta^4(\tilde{k} - k) \\ &+ i \lim_{\epsilon \rightarrow 0} \left[-i \left(\omega - i\epsilon - \frac{\tilde{p}^2}{2M_N} - \frac{\tilde{k}^2}{2M_\theta} \right)^{-1} \frac{f^*(\tilde{k}) f(k)}{h(p+k, \omega - i\epsilon)} \right] \delta^4(\tilde{p} + \tilde{k} - k - p) \end{aligned} \quad (2.30)$$

To complete the transformation to the *outgoing* spectral representation we have to calculate, according to eq. (1.24), the following quantities

$$\langle V(\tilde{p}) | \Omega_+ | \sigma, \beta \rangle_0 \quad \langle N(\tilde{p}), \theta(\tilde{k}) | \Omega_+ | \sigma, \beta \rangle_0$$

and

$$\langle V(\tilde{p}) | \Omega_+ | \sigma, \alpha \rangle_0 \quad \langle N(\tilde{p}), \theta(\tilde{k}) | \Omega_+ | \sigma, \alpha \rangle_0$$

From the second of eq. (2.12), the discussion following eq. (2.8) and the results of Appendix A, it is clear that the first two transformation matrix elements are identically zero (since $\Omega_+ |V(p)\rangle = 0$). For the first of the last pair we have

$$\langle V(\tilde{p}) | \Omega_+ | \sigma, \alpha \rangle_0 = \int d^4p \int d^4k \langle V(\tilde{p}) | \Omega_+ | N(p), \theta(k) \rangle \langle N(p), \theta(k) | \sigma, \alpha \rangle_0$$

$$\begin{aligned}
&= \int d^4p \int d^4k \delta^4(\tilde{p} - p - k) \frac{f(k)}{h(p+k, \omega - i\epsilon)} O_{k,p}^{\sigma,\alpha} \\
&= h^{-1}(\tilde{p}, \sigma - i\epsilon) \int d^4k f(k) O_{k,\tilde{p}-k}^{\sigma,\alpha} = h^{-1}(\tilde{p}, \sigma - i\epsilon) F^\alpha(\tilde{p}, \sigma)
\end{aligned} \tag{2.31}$$

where we have used Eq. (2.13) and the definition

$$F^\alpha(\tilde{p}, \sigma) \equiv \int d^4k f(k) O_{k,\tilde{p}-k}^{\sigma,\alpha} \tag{2.32}$$

Eq. (2.32) can be inverted to find $f(k)$ when given $F^\alpha(\tilde{p}, \sigma)$:

$$\begin{aligned}
\int d^4p_2 \sum_\alpha \int d\sigma O_{p_1,p_2}^{*\sigma,\alpha} F^\alpha(\tilde{p}, \sigma) &= \int d^4p_2 \sum_\alpha \int d\sigma \int d^4k f(k) O_{p_1,p_2}^{*\sigma,\alpha} O_{k,\tilde{p}-k}^{\sigma,\alpha} \\
&= \int d^4p_2 \int d^4k f(k) \delta^4(\tilde{p} - k - p_2) \delta^4(k - p_1) = f(p_1)
\end{aligned} \tag{2.32'}$$

We also obtain, in a way similar to the calculation of Eq. (2.31), the second non-vanishing matrix element

$$\begin{aligned}
&\langle N(\tilde{p}) + \theta(\tilde{k}) | \Omega_+ | \sigma, \alpha \rangle_0 = \\
&= \int d^4p \int d^4k \langle N(\tilde{p}), \theta(\tilde{k}) | \Omega_+ | N(p), \theta(k) \rangle \langle N(p), \theta(k) | \sigma, \alpha \rangle_0 = O_{\tilde{k},\tilde{p}}^{\sigma,\alpha} \\
&+ i \lim_{\epsilon \rightarrow 0} \left[-i \left(\sigma - i\epsilon - \frac{\tilde{p}^2}{2M_N} - \frac{\tilde{k}^2}{2M_\theta} \right)^{-1} f^*(\tilde{k}) h^{-1}(\tilde{p} + \tilde{k}, \sigma - i\epsilon) F^\alpha(\tilde{p} + \tilde{k}, \sigma) \right]
\end{aligned} \tag{2.33}$$

Next we turn to the calculation of the transformation to the *incoming* spectral representation. For this we have to find matrix elements of the operator Ω_- . The calculation is similar to the one given above for Ω_+ . The integral form of Ω_- is

$$\Omega_- = 1 + i \lim_{\epsilon \rightarrow 0} \int_0^{-\infty} U^\dagger(\tau) V U_0(\tau) e^{\epsilon\tau} d\tau \tag{2.34}$$

For states containing N and θ particles we have (As for Ω_+ we have that $\Omega_- |V(p)\rangle = 0$)

$$\begin{aligned}
\Omega_- |N(p), \theta(k)\rangle &= |N(p), \theta(k)\rangle + i \lim_{\epsilon \rightarrow 0} \int_0^{-\infty} d\tau U^\dagger(\tau) V U_0(\tau) e^{\epsilon\tau} a_N^\dagger(p) a_\theta^\dagger(k) |0\rangle \\
&= |N(p) + \Theta(k)\rangle - i \lim_{\epsilon \rightarrow 0} \int_0^{+\infty} d\tau U(\tau) V e^{i(\omega_N(p) + \omega_\theta(k) + i\epsilon)\tau} a_N^\dagger(p) a_\theta^\dagger(k) |0\rangle
\end{aligned} \tag{2.35}$$

Defining the transformed evolution coefficients for the linear superposition in this case to be (note the difference from eq. (2.22))

$$\begin{aligned}\tilde{A}(q, z) &= \int_0^{+\infty} d\tau e^{iz\tau} A(q, \tau) \quad \text{Im} z > 0 \\ \tilde{B}(p', k', z) &= \int_0^{+\infty} d\tau e^{iz\tau} B(p', k', \tau) \quad \text{Im} z > 0\end{aligned}\tag{2.36}$$

we find for the matrix elements of the transformation to the *incoming* spectral representation

$$\langle V(\tilde{p}) | \Omega_- | \sigma, \alpha \rangle_0 = h^{-1}(\tilde{p}, \sigma + i\epsilon) F^\alpha(\tilde{p}, \sigma)\tag{2.37}$$

and

$$\begin{aligned}\langle N(\tilde{p}), \theta(\tilde{k}) | \Omega_- | \sigma, \alpha \rangle_0 &= O_{\tilde{k}, \tilde{p}}^{\sigma, \alpha} \\ +i \lim_{\epsilon \rightarrow 0} \left[-i \left(\sigma + i\epsilon - \frac{\tilde{p}^2}{2M_N} - \frac{\tilde{k}^2}{2M_\theta} \right)^{-1} f^*(\tilde{k}) h^{-1}(\tilde{p} + \tilde{k}, \sigma + i\epsilon) F^\alpha(\tilde{p} + \tilde{k}, \sigma) \right]\end{aligned}\tag{2.38}$$

This completes the calculation of the Lax-Phillips wave operators providing the transformations to the *incoming* and *outgoing* (spectral) representations. Given these transformations it is possible in principle to construct the subspaces \mathcal{D}_\pm according to the method described in the introduction (see the discussion following eq. (1.25)). Since the orthogonality of the resulting subspaces is guaranteed only if the Lax-Phillips S-matrix satisfies the conditions (a),(b),(c) given in the introduction, we will calculate it first and then return to a discussion of the subspaces \mathcal{D}_\pm .

From Eq. (1.23) and (1.22) we have

$$\begin{aligned}_0 \langle \sigma', \alpha' | \mathbf{S} | \sigma, \beta \rangle_0 &= \langle {}_0 \sigma', \alpha' | \Omega_+^\dagger \Omega_- | \sigma, \alpha \rangle_0 = \int d^4 p {}_0 \langle \sigma', \alpha' | \Omega_+^\dagger | V(\tilde{p}) \rangle \langle V(\tilde{p}) | \Omega_- | \sigma, \alpha \rangle_0 \\ &+ \int d^4 p \int d^4 k {}_0 \langle \sigma', \alpha' | \Omega_+^\dagger | N(\tilde{p}), \theta(\tilde{k}) \rangle \langle N(\tilde{p}), \theta(\tilde{k}) | \Omega_- | \sigma, \alpha \rangle_0\end{aligned}\tag{2.39}$$

Using the expressions obtained for the wave operators Eqs. (2.32), (2.33), (2.37), (2.38) and the definition (2.31) we find that

$$\begin{aligned}_0 \langle \sigma', \alpha' | \mathbf{S} | \sigma, \alpha \rangle_0 &= \int d^4 \tilde{p} \frac{F^{\alpha' *}(\tilde{p}, \sigma') F^\alpha(\tilde{p}, \sigma)}{h(\tilde{p}, \sigma' + i\epsilon) h(\tilde{p}, \sigma + i\epsilon)} + \delta(\sigma' - \sigma) \delta_{\alpha', \alpha} \\ &+ \left[(\sigma + i\epsilon - \sigma')^{-1} \int d^4 \tilde{p} \frac{F^\alpha(\tilde{p}, \sigma) F^{\alpha' *}(\sigma', \tilde{p})}{h^{-1}(\tilde{p}, \sigma + i\epsilon)} \right] + \left[(\sigma' + i\epsilon - \sigma)^{-1} \int d^4 \tilde{p} \frac{F^{\alpha' *}(\tilde{p}, \sigma') F^\alpha(\tilde{p}, \sigma)}{h^{-1}(\tilde{p}, \sigma' + i\epsilon)} \right] \\ &+ \int d^4 \tilde{p} \int d^4 \tilde{k} \left[\left(\sigma + i\epsilon - \frac{\tilde{p}^2}{2M_N} - \frac{\tilde{k}^2}{2M_\theta} \right)^{-1} \frac{|f(\tilde{k})|^2}{h(\tilde{p} + \tilde{k}, \sigma + i\epsilon)} F^\alpha(\tilde{p} + \tilde{k}, \sigma) \right]\end{aligned}$$

$$\times \left[\left(\sigma' + i\epsilon - \frac{\tilde{p}^2}{2M_N} - \frac{\tilde{k}^2}{2M_\theta} \right)^{-1} \frac{F^{\alpha'*}(\tilde{p} + \tilde{k}, \sigma')}{h(\tilde{p} + \tilde{k}, \sigma' + i\epsilon)} \right] \quad (2.40)$$

The last term in eq. (2.40) can be put into a simpler form by the following manipulation

$$\begin{aligned} & \int d^4 \tilde{p} \left[\frac{F^\alpha(\tilde{p}, \sigma) F^{\alpha'*}(\tilde{p}, \sigma')}{h(\tilde{p}, \sigma + i\epsilon) h(\tilde{p}, \sigma' + i\epsilon)} \right] \\ & \times \int d^4 \tilde{k} \left[\left(\sigma + i\epsilon - \frac{(\tilde{p} - \tilde{k})^2}{2M_N} - \frac{\tilde{k}^2}{2M_\theta} \right)^{-1} \left(\sigma' + i\epsilon - \frac{(\tilde{p} - \tilde{k})^2}{2M_N} - \frac{\tilde{k}^2}{2M_\theta} \right)^{-1} |f(\tilde{k})|^2 \right] \\ & = \int d^4 \tilde{p} \left[\frac{F^\alpha(\tilde{p}, \sigma) F^{\alpha'*}(\tilde{p}, \sigma')}{h(\tilde{p}, \sigma + i\epsilon) h(\tilde{p}, \sigma' + i\epsilon)} \times \frac{1}{\sigma' - \sigma} (\sigma - \sigma' + h(\tilde{p}, \sigma' + i\epsilon) - h(\tilde{p}, \sigma + i\epsilon)) \right] \\ & = - \int d^4 \tilde{p} \frac{F^\alpha(\tilde{p}, \sigma) F^{\alpha'*}(\tilde{p}, \sigma')}{h(\tilde{p}, \sigma + i\epsilon) h(\tilde{p}, \sigma' + i\epsilon)} \\ & + \frac{1}{\sigma' - \sigma} \int d^4 \tilde{p} \frac{F^\alpha(\tilde{p}, \sigma) F^{\alpha'*}(\tilde{p}, \sigma')}{h(\tilde{p}, \sigma + i\epsilon)} - \frac{1}{\sigma' - \sigma} \int d^4 \tilde{p} \frac{F^\alpha(\tilde{p}, \sigma) F^{\alpha'*}(\tilde{p}, \sigma')}{h(\tilde{p}, \sigma' + i\epsilon)} \end{aligned} \quad (2.41)$$

where we have performed a partial fraction decomposition at the second step in (2.41) and used the definition Eq. (2.26) of $h(q, z)$. Combining (2.41) and (2.40) we find for the Lax-Phillips S -matrix (we use a $i\epsilon$ prescription for dealing with the singularity in (2.41))

$${}_0 \langle \sigma', \beta' | S | \sigma, \alpha \rangle_0 = \delta(\sigma' - \sigma) \left[\delta_{\alpha', \alpha} - 2\pi i \int d^4 \tilde{p} \frac{F^\alpha(\tilde{p}, \sigma) F^{\alpha'*}(\tilde{p}, \sigma')}{h(\tilde{p}, \sigma + i\epsilon)} \right] \quad (2.42)$$

We observe that in eq. (2.42) the quantity $F^{\alpha*}(\tilde{p}, \sigma)$ can be considered, for each fixed value of \tilde{p} , as a vector-valued function defined on the independent variable σ , taking its values in an auxiliary Hilbert space defined by the variables α . This observation allows us to write (see Eqs. (2.11) and (2.32))

$$F^{\alpha*}(\tilde{p}, \sigma) = (|n\rangle_{\tilde{p}, \sigma})^\alpha \quad (2.43)$$

where (for a fixed value of \tilde{p}) $(|n\rangle_{\tilde{p}, \sigma})^\alpha$ is the α component of the vector valued function $|n\rangle_{\tilde{p}, \sigma}$. With this notation we have (we suppress the auxiliary Hilbert space variables α)

$$S(\sigma) = 1 - 2\pi i \int d^4 \tilde{p} \frac{|n\rangle_{\tilde{p}, \sigma} \langle n|_{\tilde{p}, \sigma}}{h(\tilde{p}, \sigma + i\epsilon)} \quad (2.44)$$

The S -matrix given in eq. (2.44) can be simplified after further investigation into the nature of the auxiliary Hilbert space for this model, that is, after determination of the auxiliary space variables. This task is performed in the next section and we find, as expected, the foliation mentioned in the discussion following eq. (2.9) on the center of momentum variables. In section 4 we return to further analysis of the Lax-Phillips S -matrix.

3. The auxiliary Hilbert space

The characterization of the auxiliary Hilbert space of the Lax-Phillips representation of the relativistic Lee-Phriedrichs model is complete once we give exact specification of the variables α in the transformation matrix element $O_{p,k}^{\sigma,\alpha}$ of eq (2.11a) (the discussion above of the Lax-Phillips wave operators and S -matrix shows that for the solution of this problem we do not need detailed information about $O_p^{\sigma,\beta}$). Determination of these variables proceeds in two steps. We first define new independent variables $\{p, k\} \rightarrow \{P, p_{rel}\}$ by the following linear combination of p and k

$$\text{a.} \quad P = p + k \quad \text{b.} \quad p_{rel} = \frac{M_\theta p - M_N k}{M_\theta + M_N} \quad (3.1)$$

these correspond to configuration space variables

$$\text{a.} \quad X_{c.m.} = \frac{M_N x_1 + M_\theta x_2}{M_N + M_\theta} \quad \text{b.} \quad x_{rel} = x_1 - x_2$$

From eq. (2.13a) we know that

$$O_{p,k}^{\sigma,\alpha} = \delta \left(\sigma - \frac{p^2}{2M_N} - \frac{k^2}{2M_\theta} \right) \tilde{O}_{p,k}^{\sigma,\alpha}$$

This implies that

$$\sigma = \frac{p^2}{2M_N} + \frac{k^2}{2M_\theta} = \frac{P^2}{2M} + \frac{p_{rel}^2}{2\mu} \quad (3.2)$$

where $M = M_N + M_\theta$ and $\mu = (M_N M_\theta) / (M_N + M_\theta)$. We take σ and P to be independent variables; then p_{rel}^2 is fixed by:

$$p_{rel}^2 = 2\mu \left(\sigma - \frac{P^2}{2M} \right)$$

To complete the set of independent quantum numbers we then have to find a complete set of commuting operators that commute with p_{rel}^2 and P . Since p_{rel}^2 is a Casimir of the Poincaré group on the relative coordinates, the set of commuting operators contains the second Casimir of the Lorentz group and, possibly, L^2 and L_3 . We denote the set of quantum numbers corresponding to the latter three operators collectively by γ . As a consequence of this analysis we have that $\{\sigma, \alpha\} \equiv \{\sigma, P, \gamma\}$. From (2.13a) and (3.1a) it follows that

$$O_{p,k}^{\sigma,\alpha} \equiv O_{p,k}^{\sigma,P,\gamma} = \delta \left(\sigma - \frac{p^2}{2M_N} - \frac{k^2}{2M_\theta} \right) \delta^4(P - p - k) \hat{O}_{p_{rel}}^{p_{rel}^2, \gamma} \Big|_{\substack{p_{rel}^2 = 2\mu(\sigma - P^2/2M) \\ p_{rel} = (M_\theta p - M_N k)/M}} \quad (3.3)$$

Inserting this in the definition of $F_{\tilde{p},\sigma}^\alpha (\equiv F^{P,\gamma}(\tilde{p}, \sigma))$ we get

$$F^{P,\gamma}(\tilde{p}, \sigma) = \int d^4k f(k) \delta \left(\sigma - \frac{(\tilde{p} - k)^2}{2M_N} - \frac{k^2}{2M_\theta} \right) \delta^4(P - \tilde{p}) \hat{O}_{p_{rel}}^{p_{rel}^2, \gamma} \Big|_{\substack{p_{rel}^2 = 2\mu(\sigma - P^2/2M) \\ p_{rel} = M_\theta \tilde{p} / M - k}}$$

$$= \delta^4(P - \tilde{p}) \int d^4 p_{rel} f(M_\theta P/M - p_{rel}) \delta(\sigma - P^2/2M - p_{rel}^2/2\mu) \hat{O}_{p_{rel}}^{p_{rel}^2, \gamma} \quad (3.4)$$

In eq. (3.4) we see the foliation on the center of momentum discussed in the Introduction. We therefore define the following P dependent vector valued function

$$(|n\rangle_{\sigma, P})^\gamma \equiv \int d^4 p_{rel} f^*(M_\theta P/M - p_{rel}) \delta(\sigma - P^2/2M - p_{rel}^2/2\mu) (\hat{O}_{p_{rel}}^{p_{rel}^2, \gamma})^* \quad (3.5)$$

so that $F^{P, \gamma*}(\tilde{p}, \sigma) = \delta^4(P - \tilde{p}) (|n\rangle_{\sigma, P})^\gamma$. When this form of $F_{\tilde{p}, \sigma}^\alpha$ is used in eq. (2.44) for the S -matrix, taking into account Eq. (2.43), we get

$$\langle \sigma', P', \gamma' | S | \sigma, P, \gamma \rangle = \delta(\sigma' - \sigma) \delta^4(P' - P) S_P^{\gamma', \gamma}(\sigma) \quad (3.6)$$

where we define the reduced S -matrix, for a specified value P of the center of momentum 4-vector, to be

$$S_P(\sigma) \equiv 1 - 2\pi i \frac{|n\rangle_{\sigma, P} \langle n|_{\sigma, P}}{h(P, \sigma + i\epsilon)} \quad (3.7)$$

The results obtained here make it possible to represent the S -matrix in a form that admits further simplification. In order to achieve this form we consider first the completeness relation (2.14a) for the transformation matrix elements $O_{p_1, p_2}^{\sigma, P, \gamma}$. Using eq. (3.3) we obtain

$$\begin{aligned} \delta^4(p'_1 - p_1) \delta^4(p'_2 - p_2) &= \delta^4(P' - P) \delta^4(p'_{rel} - p_{rel}) = \sum_\gamma \int d\sigma \int d^4 P O_{p'_1, p'_2}^{*\sigma, P, \gamma} O_{p_1, p_2}^{\sigma, P, \gamma} \\ &= \sum_\gamma \int d\sigma \int d^4 \tilde{P} \delta(\sigma - \frac{P'^2}{2M} - \frac{p'_{rel}{}^2}{2\mu}) \delta^4(\tilde{P} - P') \hat{O}_{p'_{rel}}^{*2\mu(\sigma - \frac{\tilde{P}^2}{2M}), \gamma} \\ &\quad \times \delta(\sigma - \frac{P^2}{2M} - \frac{p_{rel}^2}{2\mu}) \delta^4(\tilde{P} - P) \hat{O}_{p_{rel}}^{2\mu(\sigma - \frac{\tilde{P}^2}{2M}), \gamma} \\ &= \delta^4(P' - P) \delta(\frac{p'_{rel}{}^2}{2\mu} - \frac{p_{rel}^2}{2\mu}) \sum_\gamma \hat{O}_{p'_{rel}}^{*p_{rel}^2, \gamma} \hat{O}_{p_{rel}}^{p_{rel}^2, \gamma} \end{aligned}$$

We therefore have a new formulation of the completeness relation

$$\delta^4(p'_{rel} - p_{rel}) = \delta(\frac{p'_{rel}{}^2}{2\mu} - \frac{p_{rel}^2}{2\mu}) \sum_\gamma \hat{O}_{p'_{rel}}^{*p_{rel}^2, \gamma} \hat{O}_{p_{rel}}^{p_{rel}^2, \gamma} \quad (3.8)$$

We construct the unit operator on the real σ axis by taking the vector valued function $|n\rangle_{\sigma, P}$ and, for each value of σ , perform an orthogonalization procedure

$$1_{H, \sigma} = \left(1_{H, \sigma} - \frac{|n\rangle_{\sigma, P} \langle n|_{\sigma, P}}{\sigma, P \langle n | n \rangle_{\sigma, P}} \right) + \frac{|n\rangle_{\sigma, P} \langle n|_{\sigma, P}}{\sigma, P \langle n | n \rangle_{\sigma, P}} \quad (3.9)$$

here $1_{H,\sigma}$ denotes the unit operator in the auxiliary Hilbert space at a point σ on the real axis. Multiplying $S_P(\sigma)$, eq. (3.7), by the unit operator eq. (3.9) we find

$$S_P(\sigma) = \left(1_{H,\sigma} - \frac{|n\rangle_{\sigma,P}\langle n|_{\sigma,P}}{\sigma,P\langle n|n\rangle_{\sigma,P}}\right) + \frac{h(P, \sigma + i\epsilon) - 2\pi i \sigma,P\langle n|n\rangle_{\sigma,P}}{h(P, \sigma + i\epsilon)} \frac{|n\rangle_{\sigma,P}\langle n|_{\sigma,P}}{\sigma,P\langle n|n\rangle_{\sigma,P}} \quad (3.10)$$

In order to simplify this expression we shall evaluate $\sigma,P\langle n|n\rangle_{\sigma,P}$ with the help of the definition, eq. (3.5), and completeness relation (3.8):

$$\begin{aligned} \sigma,P\langle n|n\rangle_{\sigma,P} &= \sum_{\gamma} \int d^4 p_{rel} \int d^4 p'_{rel} f^* \left(\frac{M_{\theta} P}{M} - p'_{rel} \right) f \left(\frac{M_{\theta} P}{M} - p_{rel} \right) \delta \left(\sigma - \frac{P^2}{2M} - \frac{p'^2_{rel}}{2\mu} \right) \\ &\quad \times \delta \left(\sigma - \frac{P^2}{2M} - \frac{p^2_{rel}}{2\mu} \right) \hat{O}_{p'_{rel}}^{*p'^2_{rel}, \gamma} \hat{O}_{p_{rel}}^{p^2_{rel}, \gamma} = \\ &= \int d^4 p_{rel} |f \left(\frac{M_{\theta} P}{M} - p_{rel} \right)|^2 \delta \left(\sigma - \frac{P^2}{2M} - \frac{p^2_{rel}}{2\mu} \right) = \int d^4 k |f(k)|^2 \delta \left(\sigma - \frac{(P-k)^2}{2M_N} - \frac{k^2}{2M_{\theta}} \right) \end{aligned} \quad (3.11)$$

We compare this result with the following expression for the jump across the cut on the real axis of the complex σ plane of $h(P, \sigma)$. Using eq. (2.26) we obtain

$$h(P, \sigma + i\epsilon) - h(P, \sigma - i\epsilon) = 2\pi i \int d^4 k |f(k)|^2 \delta \left(\sigma - \frac{(P-k)^2}{2M_N} - \frac{k^2}{2M_{\theta}} \right) \quad (3.12)$$

From eq. (3.10), (3.11) and (3.12) we see that the Lax-Phillips S-matrix becomes

$$S_P(\sigma) = \left(1_{H,\sigma} - \frac{|n\rangle_{\sigma,P}\langle n|_{\sigma,P}}{\sigma,P\langle n|n\rangle_{\sigma,P}}\right) + \frac{h(P, \sigma - i\epsilon)}{h(P, \sigma + i\epsilon)} \frac{|n\rangle_{\sigma,P}\langle n|_{\sigma,P}}{\sigma,P\langle n|n\rangle_{\sigma,P}} \quad (3.13)$$

It is possible to simplify eq. (3.13) even further by analyzing the behavior of the projection operator $P_{n,P}(\sigma)$ defined by

$$P_{n,P}(\sigma) \equiv \frac{|n\rangle_{\sigma,P}\langle n|_{\sigma,P}}{\sigma,P\langle n|n\rangle_{\sigma,P}} \quad (3.14)$$

This is done in the next section.

4. Characterization of the operator $P_{n,P}(\sigma)$

The operator valued function $P_{n,P}(\sigma)$ defined in eq. (3.14) is a projection operator for each value of σ

$$P_{n,P}(\sigma)P_{n,P}(\sigma) = P_{n,P}(\sigma) \quad (4.1)$$

It is, therefore, a bounded positive operator on the real σ axis. In order to characterize $P_{n,P}(\sigma)$ we need several definitions and results from operator theory on operator valued functions. We give these in Appendix B, where we prove that $P_{n,P}(\sigma)$ is an *outer function*¹⁸ and $|n\rangle_{\sigma,P}$ must have the form

$$|n\rangle_{\sigma,P} = g_P(\sigma)|n\rangle_P \quad (4.2)$$

where $|n\rangle_P$ is a normalized basis vector ${}_P\langle n|n\rangle_P = 1$

From eq. (4.2) and (3.11) we get

$${}_{\sigma,P}\langle n|n\rangle_{\sigma,P} = \int d^4k |f(k)|^2 \delta\left(\sigma - \frac{(P-k)^2}{2M_N} - \frac{k^2}{2M_\Theta}\right) = |g_P(\sigma)|^2 \quad (4.3)$$

Furthermore, the operator valued function $P_{n,P}(\sigma)$ becomes

$$P_{n,P}(\sigma) = \frac{|n\rangle_{\sigma,P}\langle n|_{\sigma,P}}{{}_{\sigma,P}\langle n|n\rangle_{\sigma,P}} = |n\rangle_{PP}\langle n| \quad (4.4)$$

Here the right hand side is a (constant) vector valued function of σ . This result implies a further simplification of the Lax-Phillips S -matrix

$$S_P(\sigma) = 1_H - |n\rangle_{PP}\langle n| + \frac{h(P, \sigma - i\epsilon)}{h(P, \sigma + i\epsilon)} |n\rangle_{PP}\langle n| \quad (4.5)$$

We now complete the characterization of the Lax-Phillips S -matrix $S_P(\sigma)$. The projection valued function $P_{n,P}(\sigma) = |n\rangle_{PP}\langle n|$ of equation (4.4) is to be thought of as a realization, as an operator valued function, of an operator $P_{n,P}$ which projects all functions in $L^2(-\infty, +\infty; H)$ on the subspace containing all functions taking their values in the one dimensional subspace of the auxiliary Hilbert space spanned by the single basis vector $|n\rangle_P$. If we denote the subspace of H spanned by $|n\rangle_P$ by H_1 then we have

$$P_{n,P}: L^2(-\infty, +\infty; H) \rightarrow L^2(-\infty, +\infty; H_1) \quad (4.6)$$

We denote by $P_{1-n,P}$ the operator projecting on the subspace of functions with a range in $H \ominus H_1$. We have

$$P_{1-n,P}: L^2(-\infty, +\infty; H) \rightarrow L^2(-\infty, +\infty; H \ominus H_1) \quad (4.7)$$

This operator is realized in a simple way on $L^2(-\infty, +\infty; H)$ as the operator valued function

$$T(P_{1-n,P}) = \delta(\sigma - \sigma')(1_H - |n\rangle_{PP}\langle n|) \quad (4.8)$$

(if A is an operator on the Hardy class $H_H^2(\Pi)$ then $T(A)$ is a realization of the operator A is its realization in terms of an operator valued function). It is obvious from eq. (4.4) and (4.8) that

$$L^2(-\infty, +\infty; H) = P_{n,P}L^2(-\infty, +\infty; H) \oplus P_{1-n,P}L^2(-\infty, +\infty; H) \quad (4.9)$$

In particular, if $U(\tau)$ is the operator for right shift by τ units (a left shift for $\tau < 0$) then any left shift invariant subspace $I_H^- \subset H_H^2(\Pi)$ ($H_H^2(\Pi)$ is a Hardy class of the upper half complex σ plane, see Appendix B for notation; we are working in the spectral representations of the Lax-Phillips theory) can be written as

$$I_H^- = P_{n,P}I_H^- \oplus P_{1-n,P}I_H^- \quad (4.10)$$

Now, in the spectral representation the shift operator is represented by multiplication by $e^{-i\sigma\tau}$ and we have

$$[U(\tau), P_{n,P}] = [U(\tau), P_{1-n,P}] = 0 \quad (4.11)$$

Furthermore, I_H^- is a left shift invariant subspace and therefore

$$U(\tau)I_H^- \subset I_H^- \quad \tau < 0 \quad (4.12)$$

Denote $I_{H_1}^- \equiv P_{n,P}I_H^-$, then, taking into account eq. (4.12) and (4.11) we find the following relation

$$U(\tau)I_{H_1}^- = U(\tau)P_{n,P}I_H^- = P_{n,P}U(\tau)I_H^- \subset P_{n,P}I_H^- = I_{H_1}^- \quad \tau < 0$$

or

$$U(\tau)I_{H_1}^- \subset I_{H_1}^- \quad \tau < 0 \quad (4.13)$$

We see that if I_H^- is an invariant subspace for left shifts then $I_{H_1}^- = P_{n,P}I_H^-$ is a one dimensional invariant subspace under left shifts.

In the Lax-Phillips theory the Lax-Phillips S -matrix generates a left shift invariant subspace from the Hardy class $H_H^2(\Pi)$ (this corresponds to the stability property of \mathcal{D}_-). In this case we can write

$$I_H^- = S^{LP}H_H^2(\Pi) \quad (4.14)$$

where S^{LP} is the Lax-Phillips S -matrix. From eq. (4.4) and (4.5) we see that in the case of the relativistic Lee-model we have ($S_P(\sigma)$ is the realization of S^{LP} in terms of an operator valued function)

$$[S^{LP}, P_{n,P}] = 0 \quad (4.15)$$

From eq. (4.14), (4.15) and the definition of $I_{H_1}^-$ we see that in this case

$$I_{H_1}^- = P_{n,P}I_H^- = P_{n,P}S^{LP}H_H^2(\Pi) = S^{LP}P_{n,P}H_H^2(\Pi) = S^{LP}H_{H_1}^2(\Pi)$$

where $H_{H_1}^2(\Pi) \equiv P_{n,P}H_H^2(\Pi)$. This can be rewritten as

$$I_{H_1}^- = P_{n,P}S^{LP}P_{n,P}H_{H_1}^2(\Pi) \quad (4.16)$$

From this we see that $P_{n,P}S^{LP}P_{n,P}$ generates a one dimensional left shift invariant subspace from $H_H^2(\Pi)$. From eq. (4.4) and (4.5) we get

$$T(P_{n,P}S^{LP}P_{n,P}) = \delta(\sigma - \sigma') \frac{h(P, \sigma - i\epsilon)}{h(P, \sigma + i\epsilon)} |n\rangle_{PP} \langle n| \quad (4.17)$$

This immediately implies that $h(P, \sigma - i\epsilon)/h(P, \sigma + i\epsilon)$ is a *scalar inner function*²⁴. This result comprises a complete characterization of $S_P(\sigma)$ in eq. (4.5).

A scalar inner function f can always be written as a product $f = RE$, where R is a rational inner function containing all of the zeros (and poles) of f and E is called the

singular part of f and is an inner function with no zeros²³. If we assume that in the case of the relativistic Lee-model we have only a single pole in the lower half of the complex σ plane, corresponding to a single resonance of the model, then we have in general

$$\frac{h(P, \sigma - i\epsilon)}{h(P, \sigma + i\epsilon)} = \frac{\sigma - \bar{\mu}_P}{\sigma - \mu_P} e^{if_P(\sigma)} \quad (4.18)$$

where $R = (\sigma - \tilde{\mu}_P)/(\sigma - \mu_P)$ is the rational part of the inner function and $E = e^{if_P(\sigma)}$ is the singular part.

Using eq. (4.18) in the expression for the Lax-Phillips S -matrix eq. (4.5) we note that it can be written in terms of the following product

$$S_P(\sigma) = S'_P(\sigma) M_P(\sigma) \quad (4.19)$$

where

$$\begin{aligned} S'_P(\sigma) &= 1_H - |n\rangle_{PP}\langle n| + \frac{\sigma - \bar{\mu}_P}{\sigma - \mu_P} |n\rangle_{PP}\langle n| \\ M_P(\sigma) &= 1_H - |n\rangle_{PP}\langle n| + e^{if_P(\sigma)} |n\rangle_{PP}\langle n| \end{aligned} \quad (4.20)$$

The factor $S'_P(\sigma)$ is a rational inner function containing the pole and zero of $S_P(\sigma)$ and $M_P(\sigma)$ is an inner function with no zeros. If $M_P(\sigma)$ is of exponential growth, that is, if it true that

$$|M(\sigma)| \leq e^{k|\text{Im}\sigma|}$$

for all complex σ and for some positive k , then $M_P(\sigma)$ is called a *trivial inner factor*¹.

In the framework of the Lax-Phillips theory it is possible to consider equivalent *incoming* and *outgoing* representations. These are defined in the following way¹

Definition: Two incoming (outgoing) subspaces D and D' are called *equivalent* with respect to the evolution $U(\tau)$ if there exists a real number T such that

$$U(T)D \subset D' \quad U(T)D' \subset D$$

Lax and Phillips proved that if D_- and D'_- are equivalent *incoming* subspaces then the spectral representations obtained with respect to D_- and D'_- are connected by a trivial inner factor. If a represents a vector f in H in the spectral representation with respect to D_- and a' represents the same f in the spectral representation with respect to D'_- , then there is a trivial inner factor $M_-(\sigma)$ such that

$$a'(\sigma) = M_-(\sigma) a(\sigma) \quad (4.21)$$

A similar result holds for two equivalent *outgoing* subspaces. In this case the trivial inner factor is denoted by $M_+(\sigma)$. Furthermore, if D_- , D'_- and D_+ , D'_+ are pairs of equivalent incoming and outgoing subspaces satisfying, respectively, the assumptions of the Lax-Phillips theory then the associated scattering matrices are related by

$$S' = M_+ S M_-^{-1},$$

where M_+ and M_- are the trivial inner factors which connect the spectral representations with respect to the two equivalent *outgoing* and *incoming* subspaces respectively. Lax and Phillips prove that the properties of the two scattering matrices are essentially unchanged, i.e. $S'(\sigma)$ and $S(\sigma)$ are singular at the same points in the complex plane and the generators of the semigroup in the two cases have the same spectrum. We see that in the case of equivalent problems, in the sense of the definition above, we can reduce the problem of finding the resonant state of $S_P(\sigma)$ to that of finding the resonant state of the equivalent problem for which the S -matrix is the rational factor $S'_P(\sigma)$ in eq. (4.20) and then transforming with the trivial inner factor $M_P(\sigma)$ using eq. (4.19) and (4.21) (or the analogous statement to eq. (4.21) for two equivalent *outgoing* subspaces). In the next section we find the resonant state for the case of the rational S -matrix $S'_P(\sigma)$ of eq. (4.20). The particle, i.e., N, θ or V , content of the resonant state in this case is calculated explicitly.

5. The resonant state for the rational case

In this section we shall identify the resonant state of the relativistic Lee-model in the Lax-Phillips *outgoing* translation representation for the case of a rational S -matrix of the form

$$S_P(\sigma) = 1_H - |n\rangle_{PP}\langle n| + \frac{\sigma - \bar{\mu}_P}{\sigma - \mu_P} |n\rangle_{PP}\langle n| \quad (5.1)$$

(this can also be achieved, in the same way, in the *incoming* translation representation). Having found the resonant state, we will use the transformation given in eq. (2.31) and (2.33) (or (2.37), (2.38) for the *incoming* representation), to see the particle content of the resonance.

In order to identify the resonant state we construct the projection operator P_- of the Lax-Phillips theory¹. The Hilbert space of the Lax-Phillips theory is decomposed as

$$\mathcal{H} = \mathcal{D}_- \oplus \mathcal{K} \oplus \mathcal{D}_+$$

The operator P_- is the projection into the subspace $\mathcal{K} \oplus \mathcal{D}_+$. In the *outgoing* translation representation \mathcal{D}_+ is given by $L^2(0, \infty; H)$, i.e. it is defined in a simple way by its support properties. If we obtain an expression for P_- in this representation and identify the projection onto \mathcal{D}_+ , then the remaining part will necessarily be the projection into \mathcal{K} , the subspace of the resonance (a similar procedure for the identification the resonance involves the representation of P_+ in the *incoming* translation representation and noting that in this representation \mathcal{D}_- is given in terms of support properties).

We use the fact that the subspace \mathcal{D}_- is given in the *incoming* translation representation in terms of its support properties. This allows us to write

$$P_- = \sum_{\alpha''} \int d\eta |\eta, \alpha''\rangle_{in} \theta(\eta)_{in} \langle \eta, \alpha''| = \sum_{\alpha''} \int d\eta \Omega_- |\eta, \alpha''\rangle_0 \theta(\eta)_0 \langle \eta, \alpha''| \Omega_-^\dagger \quad (5.2)$$

To represent P_- in the *outgoing* translation representation we apply eq. (1.20) and obtain

$${}_{out}\langle s, \alpha | P_- | s', \alpha' \rangle_{out} = \sum_{\alpha''} \int d\eta {}_{out}\langle s, \alpha | \Omega_- |\eta, \alpha''\rangle_0 \theta(\eta)_0 \langle \eta, \alpha'' | \Omega_-^\dagger | s', \alpha' \rangle_{out}$$

$$\begin{aligned}
&= \sum_{\alpha''} \int d\eta {}_0\langle s, \alpha | \Omega_+^\dagger \Omega_- | \eta, \alpha'' \rangle_0 \theta(\eta) {}_0\langle \eta, \alpha'' | \Omega_-^\dagger \Omega_+ | s', \alpha' \rangle_0 \\
&= \sum_{\alpha''} \int d\eta {}_0\langle s, \alpha | S | \eta, \alpha'' \rangle_0 \theta(\eta) {}_0\langle \eta, \alpha'' | S^\dagger | s', \alpha' \rangle_0
\end{aligned}$$

In this expression should to use the spectral representation of the scattering operator S and its adjoint S^\dagger . Performing the proper Fourier transforms and taking into account eq. (3.6) we get

$$\begin{aligned}
&{}_{out}\langle s, P, \gamma | P_- | s', P', \gamma' \rangle_{out} = \\
&\delta^4(P - P') \left[\sum_{\gamma''} \int d\sigma \int d\sigma' \int d\eta e^{i\sigma s} S_P^{\gamma, \gamma''}(\sigma) e^{-i\eta\sigma} \theta(\eta) e^{i\eta\sigma'} S_P^\dagger(\sigma')^{\gamma'', \gamma'} e^{-i\sigma' s'} \right] \\
&= \delta^4(P - P') \left[\frac{-i}{4\pi^2} \int d\sigma \int d\sigma' \sum_{\alpha} e^{i\sigma s} \frac{S_P^{\gamma, \gamma''}(\sigma) S_P^\dagger(\sigma')^{\gamma'', \gamma'}}{\sigma - (\sigma' + i\epsilon)} e^{-i\sigma' s'} \right] \quad (5.3)
\end{aligned}$$

The operator valued function $S_P(\sigma)$ is analytic in the upper half of the complex σ plane. The adjoint $S_P^\dagger(\sigma)$ is analytic in the lower half plane. We assume that $S_P(\sigma)$ is in the form of eq. (5.1). If the pole of $S_P(\sigma)$ is at the point μ_P , we have that the pole of $S_P^\dagger(\sigma)$ is at $\bar{\mu}_P$ and

$$S_P(\sigma) = 1 + \frac{\text{Res} S_P(\mu_P)}{\sigma - \mu_P} \quad S_P^\dagger(\sigma) = 1 + \frac{\text{Res} S_P^\dagger(\bar{\mu}_P)}{\sigma - \bar{\mu}_P} \quad \text{Im} \mu_P < 0 \quad (5.4)$$

From eq. (5.4) we see that contour integration is allowed when performing the integrals in eq. (5.3) for the various signs of s and s' . The result is

$$\begin{aligned}
&{}_{out}\langle s, P, \gamma | P_- | s', P', \gamma' \rangle_{out} = \delta^4(P - P') \\
&\times \left\{ \theta(s) \delta(s - s') \delta_{\gamma, \gamma'} + \frac{1}{2\pi} \theta(-s) \left[e^{i\mu_P s} \text{Res } S_P(\mu_P) \int d\sigma' \frac{S_P^\dagger(\sigma')}{\mu_P - (\sigma' + i\epsilon)} e^{-i\sigma' s'} \right]^{\gamma, \gamma'} = \right. \\
&\theta(s) \delta(s - s') \delta_{\gamma, \gamma'} - i\theta(-s) \theta(s') \left[e^{i\mu_P s} \text{Res } S_P(\mu_P) S_P^\dagger(\mu_P) e^{-i\mu_P s'} \right]^{\gamma, \gamma'} \\
&\left. + i\theta(-s) \theta(-s') \left[e^{i\mu_P s} \text{Res } S_P(\mu_P) \frac{\text{Res } S_P^\dagger(\bar{\mu}_P)}{\mu_P - \bar{\mu}_P} e^{-i\bar{\mu}_P s'} \right]^{\gamma, \gamma'} \right\}
\end{aligned}$$

From eq. (5.1) we have

$$\text{Res } S(\mu_P) S^\dagger(\mu_P) = 0$$

and as a consequence (in the sequel we will suppress the indices γ, γ')

$${}_{out}\langle s, P | P_- | s', P' \rangle_{out} = \delta^4(P - P')$$

$$\times \{ \theta(s) \delta(s-s') + i \theta(-s) \theta(-s') \left[e^{i\mu_P s} \text{Res } S_P(\mu_P) \frac{\text{Res } S_P^\dagger(\bar{\mu}_P)}{\mu_P - \bar{\mu}_P} e^{-i\bar{\mu}_P s'} \right] \} \quad (5.5)$$

The first term in eq. (5.5) is the projection into D_+ which, in this representation, is identified with the set of functions $L^2(0, \infty; H)$. The second term in eq. (5.5) is therefore the projection into the subspace K of the resonant state. With the help of eq. (5.1) we obtain for the second term of eq. (5.5)

$$\delta^4(P - P') [2\text{Im}\mu_P (\Theta(-s) e^{i\mu_P s} |n\rangle_P) (\theta(-s') e^{-i\bar{\mu}_P s'} {}_P\langle n|)]$$

The resonant state is the eigenstate of this projection operator. If we denote the resonant state by $|R\rangle_P$ then, for a specified value of P , we have

$${}_{out}\langle s, P, \gamma | R \rangle_{P'} = \delta^4(P - P') 2\text{Im}\mu(P) \theta(-s) e^{i\mu_P s} (|n\rangle_P)^\gamma \quad (5.6)$$

(here $2\text{Im}\mu_P$ is a normalization constant). In the spectral representation we obtain

$${}_{out}\langle \sigma, P', \gamma | R \rangle_P = \delta^4(P - P') 2i\text{Im}\mu_P \frac{(|n\rangle_P)^\gamma}{\sigma - \mu_P} \quad (5.7)$$

In order to see the particle content of the resonance we calculate the following transformation

$$\begin{aligned} \langle N(p), \theta(k) | R \rangle_P &= \sum_\alpha \int d\sigma \langle N(p), \theta(k) | \sigma, \alpha \rangle_{out} {}_{out}\langle \sigma, \alpha | R \rangle_P \\ \langle V(p) | R \rangle_P &= \sum_\alpha \int d\sigma \langle V(p) | \sigma, \alpha \rangle_{out} {}_{out}\langle \sigma, \alpha | R \rangle_P \end{aligned} \quad (5.8)$$

For the calculation of the transformations in eq. (5.8) we use Eq. (2.31), (2.33) and (5.7). Taking into account Eq. (3.3), (3.5), (3.8) and (4.2), the first term of Eq. (2.33) gives

$$\begin{aligned} 2i\text{Im}\mu_P \sum_\gamma \int d\sigma O_{k,p}^{\sigma, P, \gamma} \frac{(|n\rangle_P)^\gamma}{\sigma - \mu_P} &= 2i\text{Im}\mu_P \sum_\gamma \int d\sigma g^{-1}(\sigma) O_{k,p}^{\sigma, P, \gamma} \frac{(g(\sigma) |n\rangle_P)^\gamma}{\sigma - \mu_P} = \\ &= 2i\text{Im}\mu_P \frac{-g^{-1}(\omega_N(p) + \omega_\theta(k)) f^*(k)}{\mu_P - \frac{p^2}{2M_N} - \frac{k^2}{2M_\theta}} \delta^4(P - p - k) \end{aligned} \quad (5.9)$$

where, as before, $\omega_N(p) = p^2/2M_N$ and $\omega_\theta(k) = k^2/2M_\theta$. The second term of eq. (2.33) give the contribution

$$2i\text{Im}\mu(P) f^*(k) \delta^4(P - p - k) \int d\sigma h^{-1}(P, \sigma - i\epsilon) (\sigma - i\epsilon - \frac{p^2}{2M_N} - \frac{k^2}{2M_\theta})^{-1} \frac{g_P^*(\sigma)}{\sigma - \mu_P} \quad (5.10)$$

Assume that $g_P^*(\sigma)$ is analytic in the lower half of the complex σ plane and it possible to perform countour integretion; then, if we find that (5.10) reduces to

$$4\pi\text{Im}\mu_P f^*(k) \delta^4(P - p - k) \frac{1}{\mu_P - \frac{p^2}{2M_N} - \frac{k^2}{2M_\theta}} \frac{g_P^*(\mu_P)}{h(P, \mu_P)} \quad (5.11)$$

In this case we obtain, for the first transformation in eq. (5.8)

$$\begin{aligned} \langle N(p), \theta(k) | R \rangle_P = \\ -2i\text{Im}\mu_P \delta^4(P - p - k) \frac{f^*(k)}{\mu_P - \frac{p^2}{2M_N} - \frac{k^2}{2M_\theta}} \left(\frac{2\pi i g_P(\omega_N(p) + \omega_\theta(k)) g_P^*(\mu_P) + h(P, \mu_P)}{h(P, \mu_P) g_P(\omega_N(p) + \omega_\theta(k))} \right) \end{aligned} \quad (5.12)$$

For the second transformation in (5.8) we use eq. (2.31) and, assuming again the same analytic properties of $g_P^*(\sigma)$, we find

$$\langle V(\tilde{p}) | R \rangle_P = 2i\text{Im}\mu_P \delta^4(\tilde{p} - P) \frac{g_P^*(\mu_P)}{h(P, \mu_P)} \quad (5.13)$$

Note that if the function $g^*(\sigma)$ is a constant function then from eq. (4.3) we have $g_P(\omega_N(p) + \omega_\theta(k)) g_P^*(\mu_P) = |g_P(\sigma)|^2 = {}_{\sigma, P} \langle n | n \rangle_{\sigma, P} = |g_P(\mu_P)|^2$ in this case the numerator in eq. (5.12) vanishes and there is no N, θ component in the resonant state. In the case that the complete $S_P(\sigma)$ is a rational function, $|g^*(\sigma)|^2$ is a constant (see Appendix C), equal to $-(1/\pi)\text{Im}\mu_P$, as in the approximate Wigner-Weisskopf theory of the Lee-Friedrichs model of resonance¹⁰. In this case, $g^*(\sigma)$ is itself not necessarily a constant, but proportional to a non-trivial phase.

6. Conclusions

We have studied the application of Lax-Phillips quantum theory to a soluble relativistic quantum field theoretical model. In this model, we obtain the Lax-Phillips S -matrix explicitly as an inner function (the Lax-Phillips structure is defined pointwise on a foliation over the total energy-momentum of the system). The structure of the Lee model S -matrix (2.42) has a term with factorized numerator, corresponding to the transition matrix element of the interaction, and denominator $h(\tilde{p}, \sigma)$ which contains the zero inducing the resonance pole. The numerator factors are identified as a vector field over the complex σ plane. Foliating the S -matrix over the total energy momentum P , it takes on the form of a projection into the space complementary to the discrete subspace of the rank one potential of the model (for each point σ, P), plus an scalar inner function on the discrete subspace. The vector field on the complex extension of the spectral representation (on the singular point, it corresponds to the projection into the resonant eigenstate), is proven (Appendix B) to be independent of the spectral parameter up to a scalar multiplicative function. It then follows that the projection is in fact independent of σ . This result leads to the conclusion that the properties of the S -matrix are essentially derived from the properties of a scalar inner function.

This inner function consists of, in general, a rational factor, which contains all of the zeros and poles, and a singular factor (constructed with singular measure). If the singular factor is exponentially bounded, it is, in the terminology of Lax and Phillips¹, a trivial inner function. The application of this inner function does not change the resonance structure, but the functional form of the eigenfunctions and scattering states may be altered.

We then studied the rational case, the simplest possible model for a non-trivial Lax-Phillips theory, for which the inner function reduces to just the ratio $(\sigma - \bar{\mu}_P)/(\sigma - \mu_P)$.

We therefore see, conversely, that the simplest model for a non-trivial Lax-Phillips theory corresponds to a rank one Lee model²⁵.

For the rational case, we find the explicit form (5.7) of the resonance state in the outgoing spectral representation, of the same structure as given by Lax and Phillips¹. We furthermore give a formula for the N, θ and V components of the resonance. In the case that $g_P(\sigma)$ is independent of σ , there is no N, θ component. In Appendix C, we show (for the rational case) that the absolute value of $g_P(\sigma)$ is independent of σ , but that there is, in general, a phase which admits an N, θ component in the resonance²⁴.

The case of more than one channel (resonance) for the relativistic Lee model can be treated in a similar way. In particular, for the two channel case, providing a model, for example, of the neutral K -meson decay, one can show that the phenomenological model of Lee, Oehme and Yang¹² and Wu and Yang¹³ can be constructed directly from the more fundamental quantum Lax-Phillips theory. This work will be treated in a succeeding paper²⁶.

The study of the relativistic Lee model that we have given here appears to have all of the basic properties of a Lax-Phillips theory for a more general quantum field theory. We are presently studying the LSZ construction²⁷ in this framework.

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Appendix A.

We show that $\Omega_{\pm}|V(\tilde{p})\rangle = 0$ applying the methods used in section 2. The procedure explicitly performed for $\Omega_+|V(\tilde{p})\rangle = 0$. The result for $\Omega_-|V(\tilde{p})\rangle = 0$ is obtained in a similar way.

we start with the integral representation of the wave operator (see eq. (2.15))

$$\Omega_+ = 1 + i \lim_{\epsilon \rightarrow 0} \int_0^{+\infty} U^\dagger(\tau) V U_0(\tau) e^{-\epsilon\tau} d\tau \quad (A.1)$$

applying this operator to $|V(\tilde{p})\rangle$ we get

$$\begin{aligned} \Omega_+|V(\tilde{p})\rangle &= |V(\tilde{p})\rangle + i \lim_{\epsilon \rightarrow 0} \int_0^{+\infty} d\tau U^\dagger(\tau) V U_0(\tau) e^{-\epsilon\tau} b^\dagger(\tilde{p})|0\rangle \\ &= |V(\tilde{p})\rangle - i \lim_{\epsilon \rightarrow 0} \int_0^{-\infty} d\tau U(\tau) V e^{i(\omega_V(\tilde{p}) - i\epsilon)\tau} b^\dagger(\tilde{p})|0\rangle \end{aligned} \quad (A.2)$$

As in section 2, we want to evaluate the time evolution in the integral and perform a Laplace transform. The result of the action of the potential operator, given in eq. (2.8) to $|V(p)\rangle$, is

$$V|V(\tilde{p})\rangle = V b^\dagger(\tilde{p})|0\rangle = \int d^4k f^*(k) a_N^\dagger(\tilde{p} - k) a_\theta^\dagger(k) |0\rangle \quad (A.3)$$

A general form of a state in the sector of the fock space with $Q_1 = 1, Q_2 = 0$ is given in eq. (2.20). From eq. (A.3) we find, at time $\tau = 0$,

$$A(q, 0) = 0 \quad B(p, k, 0) = f^*(k)\delta^4(\tilde{p} - p - k) \quad (A.4)$$

Defining the Laplace transformed coefficients $\tilde{A}(q, z)$ and $\tilde{B}(p, k, z)$ as in eq. (2.23), we use eq. (2.24) and the fact that in eq. (A.4) $A(q, 0) = 0$ to obtain

$$\begin{aligned} \tilde{A}(q, z) &= \frac{i}{h(q, z)} \int d^4k f(k) \frac{B(q - k, k, 0)}{z - \frac{(q-k)^2}{2M_N} - \frac{k^2}{2M_\theta}} \\ \tilde{B}(p, k, z) &= \left(z - \frac{p^2}{2M_N} - \frac{k^2}{2M_\theta} \right)^{-1} (iB(p, k, 0) + f^*(k)\tilde{A}(p + k, z)) \end{aligned} \quad (A.5)$$

Where $h(q, z)$ is defined in eq. (2.26). Inserting in (A.5) the initial condition for $B(p, k, 0)$ from eq. (A.4) we get

$$\begin{aligned} \tilde{A}(q, z) &= i\delta^4(\tilde{p} - q) \left(\frac{z - \tilde{p}^2/2M_V}{h(q, z)} - 1 \right) \\ \tilde{B}(p, k, z) &= \left(z - \frac{p^2}{2M_N} - \frac{k^2}{2M_\theta} \right)^{-1} i f^*(k)\delta^4(\tilde{p} - p - k) \frac{z - \tilde{p}^2/2M_V}{h(\tilde{p}, z)} \end{aligned} \quad (A.6)$$

Performing the Laplace transform of eq. (2.20) implied by eq. (A.2), we use the coefficients from eq. (A.6) and evaluate the resulting expression at the point $z = \omega_V(\tilde{p}) - i\epsilon = \tilde{p}^2/2M_V - i\epsilon$. This procedure give the simple answer

$$\lim_{\epsilon \rightarrow 0} \int_0^{-\infty} d\tau U(\tau) V e^{i(\omega_V(\tilde{p}) - i\epsilon)\tau} b^\dagger(\tilde{p})|0\rangle = -ib^\dagger(\tilde{p})|0\rangle = -i|V(\tilde{p})\rangle \quad (A.7)$$

and this implies the desired result.

Appendix B.

We prove here that the vector valued function $|n\rangle_{\sigma, P}$ is necessarily of the form

$$|n\rangle_{\sigma, P} = g_P(\sigma)|n\rangle_P \quad (B.1)$$

where $|n\rangle_P$ is a fixed (for a given value of P) vector in the auxiliary Hilbert space H of the Lax-Phillips representation of the relativistic Lee-Friedrichs model.

We start with the observation made at the begining of section 4 (see Eq. (4.1) and the discussion following it) that the operator valued function $P_{n, P}(\sigma)$, defined in eq. (3.14), is a projection operator for each value of σ

$$P_{n, P}(\sigma)P_{n, P}(\sigma) = P_{n, P}(\sigma) \quad (B.2)$$

It is, therefore, a bounded positive opertor on the real σ axis (indeed, for each σ , the eigenvalue of $P_{n, P}(\sigma)$ is 1 and the eigenvector is $|n\rangle_{\sigma, P}$).

In order to proceed we need several definitions and results from the theory of operator valued functions. We denote the upper half plane of the complex σ plane by Π . If b is some separable Hilbert space, we denote by $\mathcal{B}(b)$ the set of bounded linear operators on b . We define the following sets of $\mathcal{B}(b)$ valued functions¹⁸

Definition A:

- (i) A holomorphic $\mathcal{B}(b)$ valued function $f(\sigma)$ on Π is of bounded type on Π if $\log^+ |f(\sigma)|_{\mathcal{B}(b)}$ has a harmonic majorant on Π . The class of all such functions is denoted $N_{\mathcal{B}(b)}(\Pi)$.
- (ii) If ϕ is any strongly convex function, then by $\mathcal{H}_{\phi, \mathcal{B}(b)}(\Pi)$ we mean the class of all holomorphic $\mathcal{B}(b)$ valued functions $f(\sigma)$ on Π such that $\phi(\log^+ |f(z)|_{\mathcal{B}(b)})$ has a harmonic majorant on Π .
- (iii) We define $N_{\mathcal{B}(b)}^+(\Pi) = \bigcup \mathcal{H}_{\phi, \mathcal{B}(b)}(\Pi)$, where the union is over all strongly convex functions ϕ .
- (iv) By $H_{\mathcal{B}(b)}^\infty(\Pi)$ we mean the set of all bounded holomorphic $\mathcal{B}(b)$ valued functions on Π . Here $\log^+ t = \max(\log t, 0)$ for $t > 0$ and $\log 0 = -\infty$. The sets $N_{\mathcal{B}(b)}$ and $N_{\mathcal{B}(b)}^+$ are called Nevanlinna classes and $\mathcal{H}_{\phi, \mathcal{B}(b)}(\Pi)$ is a Hardy-Orlicz class.

We will need the following theorems and definitions:

Theorem A: we have

$$H_{\mathcal{B}(b)}^\infty(\Pi) \subseteq \mathcal{H}_{\phi, \mathcal{B}(b)}(\Pi) \subseteq N_{\mathcal{B}(b)}^+(\Pi) \subseteq N_{\mathcal{B}(b)}(\Pi)$$

Definition B: Let u, v be nonzero scalar valued functions in $N^+(R)$ ($N^+(R)$ is the boundary function for a scalar Nevanlinna class function). A $\mathcal{B}(b)$ -valued function F on R is of class $\mathcal{M}(u_i, v_i)$ if $uF, vF^* \in N_{\mathcal{B}(b)}^+(R)$.

Definition C: If $A \in H_{\mathcal{B}(b)}^\infty(\Pi)$ then:

- (i) A is an *inner function* if the operator

$$T(A): f \rightarrow Af, \quad f \in H_b^2(\Pi)$$

is a *partial isometry* on $H_b^2(\Pi)$;

- (ii) A is an *outer function* if

$$\bigcup \{Af: f \in H_b^2(\Pi)\} = H_M^2(\Pi)$$

for some subspace M of b .

The main theorem which we will apply here is the following:

Theorem B: Let v be any nonzero scalar function in $N^+(R)$. If F is any nonnegative $\mathcal{B}(b)$ -valued function of class $\mathcal{M}(v, v)$ on R then

$$F = G^*G$$

on R , where G is an outer function of class $\mathcal{M}(1, v)$ on R . The factorization of F is essentially unique.

Since $P_{n,P}(\sigma)$ is a bounded operator then, from definition A(iv) and theorem A we see that

$$P_{n,P}(\sigma) \in N_{\mathcal{B}(H)}^+(\Pi).$$

where H is the auxiliary Hilbert space of the Lax-Phillips representation of the relativistic Lee-Friedrichs model, defined by the variables γ in eq. (3.3) (or eq. (3.5),(3.6)). Furthermore, the projection operator $P_{n,P}(\sigma)$ satisfies $(P_{n,P}(\sigma))^* = P_{n,P}(\sigma)$ and, from definition C we immediately have

$$P_{n,P}(\sigma) \in \mathcal{M}(1, 1)$$

We can apply theorem B with the result that there is a unique decomposition of $P_{n,P}(\sigma)$

$$P_{n,P}(\sigma) = G^*G = (P_{n,P}(\sigma))^*P_{n,P}(\sigma) = P_{n,P}(\sigma)P_{n,P}(\sigma)$$

and that $G = P_{n,P}(\sigma)$ is an *outer function*. Denote by $P_{n,p}$ the operator on $H_H^2(\Pi)$ for which the realization as an operator valued function is $P_{n,p}(\sigma)$. From definition C(ii) we therefore have

$$\left\{ \bigcup P_{n,P}f : f \in H_H^2(\Pi) \right\} = H_M^2(\Pi)$$

where M is a subspace of the auxiliary Hilbert space H . If $|f\rangle_\sigma$ is the vector valued function corresponding to some $f \in H_H^2(\Pi)$ then we can write this explicitly as

$$\left\{ \bigcup \frac{|n\rangle_{\sigma,P} \langle n|_{\sigma,P}}{\sigma, P \langle n|n \rangle_{\sigma,P}} |f\rangle_\sigma : |f\rangle_\sigma = f \in H_H^2(\Pi) \right\} = H_M^2(\Pi)$$

Since $P_{n,P}(\sigma)$ is a projection operator for each σ then its range for each σ is a vector proportionl to $|n\rangle_{\sigma,P}$. Define

$$m = \left\{ |v\rangle : |v\rangle \in b \text{ and } |v\rangle = |n\rangle_{\sigma,P} \text{ for some } \sigma \in \Pi \right\}$$

Denote by M the subspace of b spanned by the vectors in m . Clearly the range of $P_{n,P}$ lies in M and there is no smaller subspace of b that contains the range of $P_{n,P}$. Therefore we have

$$\left\{ \bigcup P_{n,P}f : f \in H_H^2(\Pi) \right\} \subseteq H_M^2(\Pi)$$

If the dimension of M is greater then one, a function $f \in H_M^2(\Pi)$ can always be found that is not in $\left\{ \bigcup P_{n,P}f : f \in H_H^2(\Pi) \right\}$ (for example, take a specific value $\sigma = \sigma_0$ and define ,for some Hardy class function $g(\sigma)$, $j(\sigma) = g(\sigma)|n\rangle_{\sigma_0,P}$; clearly $j \in H_M^2(\Pi)$ but $j \notin \left\{ \bigcup P_{n,P}f : f \in H_H^2(\Pi) \right\}$). It is therefore true that

$$\left\{ \bigcup P_{n,P}f : f \in H_H^2(\Pi) \right\} \subset H_M^2(\Pi), \quad \text{Dim } M > 1.$$

Therefore $P_{n,P}$ cannot be an outer function unless $\text{Dim } M = 1$. If $\text{Dim } M = 1$ we have necessarily that

$$|n\rangle_{\sigma,P} = g_P(\sigma)|n\rangle_P.$$

Appendix C.

We prove that in the case of a rational S -matrix of the form

$$S_P(\sigma) = 1_H - |n\rangle_{PP}\langle n| + \frac{\sigma - \bar{\mu}_P}{\sigma - \mu_P} |n\rangle_{PP}\langle n| \quad (C.1)$$

the analytic continuation to the complex σ plane of the function defined on the real σ axis by

$$J_P(\sigma) \equiv {}_{\sigma,P}\langle n|n\rangle_{\sigma,P} = |g_P(\sigma)|^2 \quad (C.2)$$

is a constant.

We have proved in section 4 that the Lax-Phillips S -matrix in the relativistic Lee-Friedrichs model has the general form (see Eq. (4.5))

$$S_P(\sigma) = 1_H - |n\rangle_{PP}\langle n| + \frac{h(P, \sigma - i\epsilon)}{h(P, \sigma + i\epsilon)} |n\rangle_{PP}\langle n| \quad (C.3)$$

The function $h(P, \sigma - i\epsilon)/h(P, \sigma + i\epsilon)$ should then be a scalar inner function. From Eq. (3.12), (4.3) and (C.2) we see that one can write this function as

$$\frac{h(P, \sigma - i\epsilon)}{h(P, \sigma + i\epsilon)} = \frac{h(P, \sigma + i\epsilon) - 2\pi i J_P(\sigma)}{h(P, \sigma + i\epsilon)} = 1 - 2\pi i \frac{J_P(\sigma)}{h(P, \sigma + i\epsilon)}. \quad (C.4)$$

We define

$$W_P(\sigma) \equiv \text{Re } h(P, \sigma \pm i\epsilon) = \sigma - \frac{P^2}{2M_V} - \mathcal{P} \int d^4k \frac{|f(k)|^2}{\sigma - \frac{(P-k)^2}{2M_N} - \frac{k^2}{2M_\theta}} \quad (C.5)$$

Where \mathcal{P} is the symbol for the principle part of the integral. Using this definition, the functions $h(P, \sigma \pm i\epsilon)$ can be written as

$$h(P, \sigma \pm i\epsilon) = W_P(\sigma) \pm i\pi J_P(\sigma) \quad (C.6)$$

We have

$$\frac{h(P, \sigma - i\epsilon)}{h(P, \sigma + i\epsilon)} = \frac{W_P(\sigma) - i\pi J_P(\sigma)}{W_P(\sigma) + i\pi J_P(\sigma)} \quad (C.7)$$

In the upper half of the complex σ plane we require that $S_P(\sigma)$ satisfies properties (a), (b) and (c) listed in the introduction ($S_P(\sigma)$ is then an operator inner function). That property (c) is satisfied one infers from Eq. (C.7). If we require in (C.4) that the function $J_P(\sigma)/h(P, \sigma)$ is analytic in the upper half of the complex σ plane then the Lax-Phillips S -matrix in Eq. (C.3) satisfies also property (a). To satisfy property (b) we require that this function is bounded in the upper half plane,

$$\left| \frac{J_P(\sigma)}{h(P, \sigma)} \right| \leq M_1, \quad (\text{Im } \sigma > 0) \quad (C.8)$$

for some positive bound M_1 .

The adjoint of the S -matrix $S_P^\dagger(\sigma)$ given (σ real) by

$$S_P^\dagger(\sigma) = 1_H - |n\rangle_{PP}\langle n| + \frac{h(P, \sigma + i\epsilon)}{h(P, \sigma - i\epsilon)} |n\rangle_{PP}\langle n| \quad (C.9)$$

is a map from the *outgoing* spectral representation to the *incoming* spectral representation. It has the properties (a'), (b') and (c') obtained from (a), (b) and (c) by replacing everywhere $\text{Im}\sigma > 0$ by $\text{Im}\sigma < 0$ (the Fourier transform of this operator valued function generates right translation invariant subspaces of $L^2(0, \infty; H)$). We see from Eq. (C.7) and (C.9) that property (c') is satisfied. Furthermore, we have

$$\frac{h(P, \sigma + i\epsilon)}{h(P, \sigma - i\epsilon)} = \frac{h(P, \sigma - i\epsilon) + 2\pi i J_P(\sigma)}{h(P, \sigma - i\epsilon)} = 1 + 2\pi i \frac{J_P(\sigma)}{h(P, \sigma - i\epsilon)} \quad (C.10)$$

In order to satisfy property (a') we require that $J_P(\sigma)/h(P, \sigma)$ is analytic in the lower half of the complex σ plane. Property (b') is satisfied if this function is also bounded there, *i.e.* if

$$\left| \frac{J_P(\sigma)}{h(P, \sigma)} \right| \leq M_2 \quad \text{Im}\sigma < 0 \quad (C.11)$$

for some positive bound M_2 .

The function $h(P, \sigma)$, as explicitly defined in (2.26) (these are actually two different functions in the lower and upper half plane; see for example Eq. (3.12)), has no poles either in the upper half plane or in the lower half plane. By the requirements of boundedness Eq. (C.8) and (C.11) we have that $J_P(\sigma)$ has no poles in the complex σ plane. In this case $J_P(\sigma)$ is an *entire function*.

In the case of the rational Lax-Phillips S -matrix of Eq. (C.1) we have

$$\frac{h(P, \sigma - i\epsilon)}{h(P, \sigma + i\epsilon)} = \frac{\sigma - \bar{\mu}_P}{\sigma - \mu_P} \quad (C.12)$$

This immediately implies the following condition (here σ is real)

$$\text{Im}((\sigma - \mu_P)h(P, \sigma - i\epsilon)) = 0 \quad (C.13)$$

and, using Eq. (C.6), we have

$$-\pi(\sigma - \text{Re}\mu_P)J_P(\sigma) = \text{Im}\mu_P W_P(\sigma) = \text{Im}\mu_P(h(P, \sigma - i\epsilon) + i\pi J_P(\sigma))$$

this can be written in a more compact form

$$-\pi(\sigma - \mu_P)J_P(\sigma) = h(P, \sigma - i\epsilon)\text{Im}\mu_P \quad (C.14)$$

This relation can be analytically continued to the lower half plane. In the limit $\sigma \rightarrow \infty$ (in the lower half plane) we find

$$\lim_{\sigma \rightarrow \infty} J_P(\sigma) = -\frac{1}{\pi} \text{Im}\mu(P) \quad \text{Im}\sigma < 0 \quad (C.15)$$

The complex conjugate of (C.14) can be analytically continued to the upper half plane and find again that

$$\lim_{\sigma \rightarrow \infty} J_P(\sigma) = -\frac{1}{\pi} \text{Im} \mu(P) \quad \text{Im} \sigma > 0 \quad (\text{C.16})$$

Since $J_P(\sigma)$ is an entire function, Eq. (C.15) and (C.16) imply that it is a constant

$$J_P(\sigma) = |g_P(\sigma)|^2 = -\frac{1}{\pi} \text{Im} \mu_P \quad (\text{C.17})$$

This result does not imply that the numerator of (5.12) vanishes. We have shown that in the rational case, the absolute value of $g_P(\sigma)$ is constant, but it may have a nontrivial phase.

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